HYDROMECHANICS.

PART I.

HYDROSTATICS.

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A TREATISE

ON

HYDROMECHANICS.

PART I.

HYDROSTATICS.

BY

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PREFACE.

THE present Treatise is a reproduction, with considerable alterations and additions, of the first part of a treatise on Hydrostatics and Hydrokinetics, the third edition of which was published in 1877.

I have thought it advisable, for the convenience of students, and for other reasons, to issue the part on the Equilibrium of Fluids in a separate volume, at any rate for the present, and to reserve the discussion of the Motion of Fluids for another volume.

This I hope, if time and health permit, to have ready early next year.

The instalment of Hydrostatics, which I now offer to the student, is intended to cover the ground ranged over, so far as the subject is concerned, in the Examination in Schedule II. for the Mathematical Tripos, and to serve as a stepping-stone in the advance to the higher regions of Hydrokinetics, and its applications to the theory of sound, and the oscillations of liquid waves.

A more careful discussion of Curves of Buoyancy, the extension of the chapter on Tension, the addition of a chapter on Capillarity, and the insertion of some fresh examples, taken from recent examination papers, with other alterations and additions, will, I hope, render the present edition interesting and useful to mathematical students.

W. H. BESANT.

October 28, 1882.

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HYDROSTATICS.

CHAPTER I.

1. WE learn from common experience that such substances as air and water are characterised by the ease with which portions of their mass can be removed, and by their extreme divisibility. These properties are illustrated by various common facts; if, for instance, we consider the ease with which fluids can be made to permeate each other, the extreme tenuity to which one fluid can be reduced by mixture with a large portion of another fluid, the rarefaction of air which can be effected by means of an air-pump, and other facts of a similar kind, it is clear that, practically, the divisibility of fluid is unlimited: we find, moreover, that in separating portions of fluids from each other, the resistance offered to the division is very slight, and in general almost inappreciable. By a generalization from such observations, the conception naturally arises of a substance possessing in the highest degree these properties, which exist, in a greater or less degree, in every fluid with which we are acquainted, and hence we are led to the following

Definition of a Perfect Fluid.

2. *A perfect fluid is an aggregation of particles which yield at once to the slightest effort made to separate them from each other.*

If then an indefinitely thin plane be made to divide such a fluid in any direction, no resistance will be offered to the division, and the pressure exerted by the fluid on the plane

will be entirely normal to it; that is, a perfect fluid is assumed to have no "viscosity," no property of the nature of friction.

The following fundamental property of a fluid is therefore obtained from the above definition.

The pressure of a perfect fluid is always normal to any surface with which it is in contact.

As a matter of fact, all fluids do more or less offer a resistance to separation or division, but, just as the idea of a rigid body is obtained from the observation of bodies in nature which only change form slightly on the application of great force, so is the idea of a perfect fluid obtained from our experiences of substances which possess the characteristics of extremely easy separability and apparently unlimited divisibility.

The following definition will include fluids of all degrees of viscosity.

A fluid is an aggregation of particles which yield to the slightest effort made to separate them from each other, if it be continued long enough.

Hence it follows that, in a viscous fluid at rest, there can be no tangential action, or shearing stress, and therefore, as in the case of a perfect fluid,

The pressure of a fluid at rest is always normal to any surface with which it is in contact.

Thus all propositions in Hydrostatics are true for all fluids whatever be the viscosity.

It is in Hydrodynamics that we are limited to the consideration of *perfect* fluids.

3. Fluids are divided into Liquids and Gases; the former, such as water and mercury, are not sensibly compressible, except under very great pressures; the latter are easily compressible, and expand freely if permitted to do so.

Hence the former are sometimes called inelastic, and the latter elastic fluids.

4. Fluids are acted upon by the force of gravity in the same way as solids; with regard to liquids this is obvious; and that air has weight can be shewn directly by weighing a closed vessel, exhausted as far as possible: moreover, the phenomena of the tides shew that fluids are subject to the attractive forces of the sun and moon as well as of the earth, and it is assumed, from these and other similar facts, that fluids of all kinds are subject to the law of gravitation, that is, that they attract, and are attracted by, all other portions of matter, in accordance with that law.

Measure of the Pressure of Fluids.

5. Consider a mass of fluid at rest under the action of any forces, and let A be the area of a plane surface exposed to the action of the fluid, that is, in contact with it, and P the force which is required to counterbalance the action of the fluid upon A. If the action of the fluid upon A be uniform, then $\frac{P}{A}$ is the pressure on each unit of the area A. If the pressure be not uniform, it must be considered as varying continuously from point to point of the area A, and if ϖ be the force on a small portion α of the area about a given point, then $\frac{\varpi}{\alpha}$ will approximately express the *rate* of pressure over α. When α is indefinitely diminished let $\frac{\varpi}{\alpha}$ ultimately $= p$, then p is defined to be the measure of the pressure at the point considered, p being the force which would be exerted on an unit of area, if the rate of pressure over the unit were uniform and the same as at the point considered.

The force upon any small area α about a point, the pressure *at* which is p, is therefore $p\alpha + \gamma$, where γ vanishes ultimately in comparison with $p\alpha$ when α (and consequently $p\alpha$) vanishes.

6. *The pressure at any point of a fluid at rest is the same in every direction.*

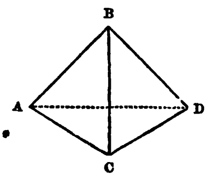
This is the most important of the characteristic properties

of a fluid; it can be deduced from the fundamental property of a fluid in the following manner:

If we consider the equilibrium of a small tetrahedron of fluid, we observe that the pressures on its faces, and the impressed force on its mass, form a system of equilibrating forces.

The former forces depending on the areas of the faces vary as the square, and the latter depending on the volume and density varies as the cube of one of the edges of the solid, which is considered to be homogeneous, and therefore supposing the solid indefinitely diminished, while it retains always a similar form, the latter force vanishes in comparison with the pressures on the faces; and these pressures consequently form of themselves a system of forces in equilibrium.

Let p, p' be the rates of pressure on the faces ABC, BCD, and resolve the forces parallel to the edge AD; then, since the projections of the areas ABC, BCD on a plane perpendicular to AD are the same (each equal to a suppose), we have ultimately,



$$pa = p'a,$$

or $$p = p'.$$

And similarly it may be shewn that the pressures on the other two faces are each equal to p or p'.

As the tetrahedron may be taken with its faces in any direction, it follows that the pressure at a point is the same in every direction.

7. The following proof of the foregoing proposition is taken from Cauchy's *Exercices**.

Let P and Q be two points in a fluid at a finite distance from each other; about PQ as axis describe a cylinder of very small radius, draw a plane through Q perpendicular to QP, draw any plane through P, and consider the equilibrium of the mass PQ.

* *Seconde Année*, 1827, p. 23.

The pressures on its ends and on its curved surface, and the impressed forces which act upon it, form a system of balancing forces.

Let p, p' be the pressures at Q and P, α the area of the section Q of the cylinder, and α' of the section P; then the pressure $p'\alpha'$ on the end P,

resolved parallel to the axis of the cylinder, is equal to $p'\alpha$, and therefore

$$p'\alpha - p\alpha = \text{the impressed force, resolved parallel to } QP.$$

Now whatever be the direction of the plane through P, this impressed force, when the radius of the cylinder is indefinitely diminished, is ultimately equal to the impressed force on the portion QP of the cylinder cut off by a plane through P perpendicular to the axis *, that is, to

$$\int_0^{PQ} f\rho\alpha\, dx,$$

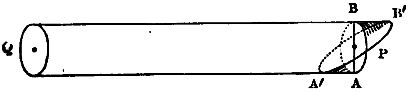
where mf is the force on a particle m of the fluid at a dis tance x from Q. Hence

$$p' = p + \int_0^{PQ} \rho f\, dx,$$

or p' is constant for all positions of the plane through P.

* The following considerations may complete this part of the proof:

Let AB, $A'B'$ be the two planes through P; $\rho\rho'$ the mean densities of APA', BPB'; and f, f' the accelerations of the forces which are acting on these portions of fluid.



Then the difference of the forces on QAB and $QA'B'$ (the volumes of which are equal)

$$= \text{the difference of the forces on } APA' \text{ and } BPB'$$
$$= (\rho'f' - \rho f) \,.\, \text{vol} \,.\, APA'$$
$$= \delta(\rho f) \,.\, \frac{2}{3\pi}\, \alpha AA',$$

and therefore $p' = p + \int_0^{QP} \rho f dx + \frac{2}{3\pi} AA' \,.\, \delta(\rho f)$.

The forces being continuous, the last term is obviously evanescent compared with the other quantities in the equation, and p' is therefore constant.

Transmission of Fluid Pressure.

8. *Any pressure, or additional pressure, applied to the surface or to any other part of a liquid at rest, is transmitted equally to all parts of the liquid.*

This property of liquids is a direct result of experiment, and, as such, is sometimes assumed. It is however deducible from the definition of a fluid.

Let P be a point in the surface of a liquid at rest, and Q any other point in the liquid; about the straight line PQ describe a cylinder, of very small radius, bounded by the surface at P and by a plane through Q, perpendicular to QP.

If the pressure at P be increased by p, the additional force on the cylinder, resolved in the direction of its axis, is $p\alpha$, α being the area of the section of the cylinder perpendicular to its axis, and this must be counteracted by an equal force $p\alpha$ at Q in the direction QP, since the pressure of the liquid on the curved surface is perpendicular to the axis. The pressure at Q is therefore increased by p.

If the straight line PQ do not lie entirely in the liquid, P and Q can be connected by a number of straight lines, all lying in the liquid, and a repetition of the above reasoning will shew that the pressure p is transmitted, unchanged, to the point Q.

9. In consequence of this property, a mass of liquid can be used as a 'machine' for the purpose of multiplying power.

Thus, if in a closed vessel full of water two apertures be made and pistons A, A' fitted in them, any force P applied to one piston must be counteracted by a force P' on the other piston, such that $P' : P$ in the ratio of the area $A' : A$, for the increased rate of pressure at every point of A is transmitted to every point of A', and the force upon A' depends therefore upon its area *.

The action between the two is analogous to the action of a lever, and it is clear that by increasing A' and diminishing A, we can make the ratio $P' : P$ as large as we please.

* Bramah's Press is an instance of the practical use of this property of liquids.

10. The pressure of a gaseous fluid is found to depend upon its density and temperature, as well as upon the nature of the fluid itself.

When the temperature is constant, experiment shews that the pressure varies inversely as the space occupied by the fluid, that is, directly as its density.

This law was first stated by Boyle, but it is a consequence of the more general law that the pressure of a mixture of gases that do not act chemically on each other is the sum of the pressures the gases would exert if they filled the containing vessel separately. For doubling the quantity of gas in the vessel would double the pressure, and a similar proportionate change of pressure would take place for any other change of quantity.

Hence if ρ be the density of a certain quantity of a gaseous fluid, and p its pressure, then, as long as the temperature remains the same,

$$p = k\rho,$$

where k is a constant, to be determined experimentally for the fluid at a given temperature.

If v be the volume of the gas at the pressure p, and v' the volume at the pressure p',

$$pv = p'v',$$

or pv is constant for a given temperature.

11. The *Elasticity* of a fluid is measured by the ratio of a small increase of pressure to the cubical compression produced by it.

If v be the volume, the small cubical compression is $\dfrac{dv}{v}$, and the measure of the elasticity is

$$-v\frac{dp}{dv}.$$

In the case of a gas pv is constant,

$$\text{and } \therefore p + v\frac{dp}{dv} = 0,$$

so that the measure of the elasticity is equal to that of the pressure.

Measures of Weight, Mass, and Density.

12. The weight, mass, and density of a fluid are measured in the same way as for solid bodies.

If W be the weight of a mass M of fluid, then, in accordance with the usual conventions which define the units of mass and force,
$$W = Mg.$$

If V be the volume of the mass M of fluid of density ρ, then
$$M = \rho V,$$
$$\text{and } \therefore\ W = g\rho V.$$

For the standard substance, $\rho = 1$, and therefore the unit of volume of the standard substance is the unit of mass.

13. In the previous articles no account has been taken of fluids in which the density is variable; but it is easy to conceive the density of a mass of liquid varying continuously from point to point, and it will be hereafter found that a mass of elastic fluid, at rest under the action of gravity, and having a constant temperature throughout, is necessarily heterogeneous: the density at a point of a fluid must therefore be measured in the same way as the pressure at a point, or any other continuously varying quantity.

Measure of the density at any point of a heterogeneous fluid.

Let m be the mass of a volume v of fluid enclosing a given point, and suppose ρ the density of a homogeneous fluid such that the mass of a volume v is equal to m, or such that
$$m = \rho v;$$
then ρ may be defined as the mean density of the portion v of the heterogeneous fluid, and the ultimate value of ρ when v is indefinitely diminished, supposing it always to enclose the point, is the density of the fluid at that point.

14. *To find the work done in compressing a gas.*

Let v be the volume of a gas at the pressure p, ds an element of the surface of the vessel containing it, and dn an element of the normal to ds drawn inwards.

Then the work done in a small compression

$$= p\Sigma dsdn = -pdv,$$

and the work done in compressing from V to V'

$$= -\int pdv = -\int \frac{Cdv}{v}, \text{ if } pv = C,$$

$$= C \log \frac{V}{V'} = pv \log \frac{V}{V'}.$$

EXAMPLES.

(In these Examples g is taken to be 32, when a foot and a second are units.)

1. In a Hydraulic Press the diameter of the ram is nine inches and of the plunger of the pump is one inch; the length of the pump-lever is three feet, and the distance of the point of attachment of the plunger from the fulcrum is nine inches. If a force of 15 lbs. weight be applied at the end of the lever, find the force exerted by the ram of the press.

2. $ABCD$ is a rectangular area subject to fluid pressure; AB is a fixed line, and the pressure on the area is a given function (P) of the length BC (x); prove that the pressure at any point of CD is $\frac{dP}{adx}$, where $a = AB$.

If A be a fixed point, and AB, AD fixed in direction, and if $AB = x$ and $AD = y$, the pressure at $C = \frac{d^2P}{dxdy}$.

3. In the equation $W = g\rho V$, if the unit of force be 100 lbs. weight, the unit of length 2 feet, and the unit of time $\frac{1}{4}$th of a second, find the density of water.

4. If a minute be the unit of time, and a yard the unit of space, and if 15 cubic inches of the standard substance contain 25 oz., determine the unit of force.

5. In the equation, $W = g\rho V$, the number of seconds in the unit of time is equal to the number of feet in the unit of length, the unit of force is 750 lbs. weight, and a cubic foot

of the standard substance contains 13500 ounces; find the unit of time.

6. A velocity of 4 feet per second is the unit of velocity; water is the standard substance and the unit of force is 125 lbs. weight; find the units of time and length.

7. The number expressing the weight of a cubic foot of water is $\frac{1}{10}$th of that expressing its volume, $\frac{1}{8}$th of that expressing its mass, and $\frac{1}{100}$th of the number expressing the work done in lifting it 1 foot. Find the units of length, mass, and time.

8. Mercury (density 13·6) is the standard substance, 4 inches is the unit of length, and the acceleration due to gravity is denoted by 54: find the unit of force in absolute foot-pound-second units.

9. If a feet and b seconds be the units of space and time, and the density of water the standard density, find the relation between a and b in order that the equation, $W = g\rho V$, may give the weight of a substance in pounds.

10. A velocity of 8 feet per second is the unit of velocity, the unit of acceleration is that of a falling body, and the unit of mass is a ton; find the density of water.

11. The density at any point of a liquid, contained in a cone having its axis vertical and vertex downwards, is greater than the density at the surface by a quantity varying as the depth of the point. Shew that the density of the liquid when mixed up so as to be uniform will be that of the liquid originally at the depth of one-fourth of the axis of the cone.

12. The density of a fluid varies from point to point; considering directions proceeding from a given point, prove that the density varies most rapidly along the normal to the surface of equal density containing the point; and of directions in the tangent plane to this surface, the tangents to its principal sections are those in which the rate of variation of density is greatest and least.

CHAPTER II.

THE CONDITIONS OF THE EQUILIBRIUM OF FLUIDS.

15. TAKING the most general case, suppose a mass of fluid, elastic or non-elastic, homogeneous or heterogeneous, to be at rest under the action of given forces, and let it be required to determine the conditions of equilibrium, and the pressure at any point.

Let x, y, z be the co-ordinates referred to rectangular axes, of any point P in the fluid, and let Q be a point near it, so taken that PQ is parallel to the axis of x.

Take $x + \delta x$, y, z, as the co-ordinates of Q; about PQ describe a small prism or cylinder bounded by planes perpendicular to PQ.

Let α be the area of the section of the cylinder perpendicular to its axis, p the pressure at P, and $p + \delta p$ the pressure at Q.

Then, α being very small, the pressure at any point of the plane P will be very nearly equal to p, and the pressure upon it will therefore be

$$(p + \gamma)\,\alpha,$$

where γ vanishes in comparison with p when α is indefinitely diminished.

We can therefore consider α so small that γ may be neglected in comparison with p, and the pressure on the end P of the cylinder may be taken equal to $p\alpha$, and similarly the pressure on the end Q equal to

$$(p + \delta p)\,\alpha.$$

If ρ be the mean density of the cylinder PQ, its mass $=\rho\alpha\delta x$, and $X\rho\alpha\delta x$ will represent the force on PQ parallel to its axis, if $X\delta m$, $Y\delta m$, $Z\delta m$ be the components of the forces acting on a particle δm of fluid at the point xyz.

Hence, for the equilibrium of PQ,

$$(p+\delta p)\,\alpha - p\alpha = X\rho\alpha\delta x,$$

or $$\delta p = \rho X\delta x.$$

Proceeding to the limit when δx, and therefore δp, is indefinitely diminished, ρ will be the density at P, and we obtain

$$\frac{dp}{dx} = \rho X^{*}.$$

By a similar process,

$$\frac{dp}{dy} = \rho Y,$$

$$\frac{dp}{dz} = \rho Z.$$

But $$dp = \frac{dp}{dx}dx + \frac{dp}{dy}dy + \frac{dp}{dz}dz;$$

$$\therefore\ dp = \rho\,(Xdx + Ydy + Zdz)\ldots\ldots\ldots\ldots(\alpha),$$

the equation which determines the pressure.

16. It is therefore an essential condition of equilibrium that $\rho\,(Xdx + Ydy + Zdz)$ should be a perfect differential of some function $f(x, y, z)$; and

$$\left.\begin{aligned} \therefore\ \frac{d}{dy}(\rho Z) &= \frac{d}{dz}(\rho Y) \\ \frac{d}{dz}(\rho X) &= \frac{d}{dx}(\rho Z) \\ \frac{d}{dx}(\rho Y) &= \frac{d}{dy}(\rho X) \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(\beta),$$

* In the above proof, α is taken so small that its linear dimensions may be neglected in comparison with δx; that is, the change in p, corresponding to a change δx in x, is considered, undisturbed by any alterations in y and z.

from which by differentiating, multiplying the equations respectively by X, Y, and Z, and adding, we obtain

$$X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right)=0\ldots.(\gamma),$$

a necessary condition of equilibrium.

The geometrical interpretation of this equation is that the lines of force,

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

can be intersected orthogonally by a system of surfaces.

17. *Homogeneous Liquids.* If the fluid be homogeneous and incompressible, $Xdx+Ydy+Zdz$ must be a perfect differential in order that equilibrium may be possible.

In other words, the system of forces must be a conservative system, and the forces can be represented by the space-variations of a potential function.

We then have, if V be the potential function,

$$dp=-\rho dV,$$

and $$\therefore \frac{p}{\rho}+V=C.$$

18. If, for instance, the forces tend to or from fixed centres and are functions of the distances from those centres, we have

$$X=\Sigma\left\{\phi(r)\frac{x-a}{r}\right\},\quad Y=\Sigma\left\{\phi(r)\frac{y-b}{r}\right\},\quad Z=\Sigma\left\{\phi(r)\frac{z-c}{r}\right\},$$

where (a, b, c) are co-ordinates of the centre to which the force $\phi(r)$ tends.

Now $$r^2=(x-a)^2+(y-b)^2+(z-c)^2,$$

$$\therefore Xdx+Ydy+Zdz=\Sigma\phi(r)\,dr,$$

and $$dp=\rho\Sigma\phi(r)\,dr.$$

In this case, since

$$\frac{dX}{dy} = \Sigma \left\{\phi'(r)\frac{x-a}{r}\frac{y-b}{r} - \phi(r)\frac{x-a}{r^2}\frac{y-b}{r}\right\},$$

$$\text{and } \frac{dY}{dx} = \Sigma \left\{\phi'(r)\frac{y-b}{r}\frac{x-a}{r} - \phi(r)\frac{y-b}{r^2}\frac{x-a}{r}\right\},$$

it is obvious that the equation (γ) is always satisfied, but it is not to be inferred that the equilibrium of a heterogeneous fluid is always possible with such a system of forces.

When the density is constant, the equations (β) become

$$\frac{dX}{dy} = \frac{dY}{dx}, \quad \frac{dZ}{dy} = \frac{dY}{dz}, \quad \frac{dX}{dz} = \frac{dZ}{dx},$$

which are in this case always satisfied, and therefore the equilibrium of a homogeneous fluid under the action of such forces is always possible.

19. *Elastic Fluids.* When the fluid is elastic, an additional condition is introduced, for, if the temperature be constant,

$$p = k\rho;$$

$$\therefore \frac{dp}{p} = \frac{1}{k}(Xdx + Ydy + Zdz) \;\ldots\ldots\ldots\; (\delta).$$

If the forces are derivable from a potential V, *i.e.* if $Xdx + Ydy + Zdz$ be a perfect differential $-dV$,

$$k\frac{dp}{p} = -dV,$$

$$\therefore k \log \frac{p}{C} = -V,$$

$$\text{or } p = C\epsilon^{-\frac{V}{k}}, \text{ and } \rho = \frac{C}{k}\epsilon^{-\frac{V}{k}}.$$

When the forces tend to fixed centres and are functions of the distances, Art. (18), this equation takes the form

$$k\frac{dp}{p} = \Sigma\phi(r)\,dr,$$

and p can be determined.

If the temperature be variable, the relation between the pressure, density, and temperature is found to be

$$p = k\rho\,(1 + \alpha t),$$

where t is the temperature, measured by a Centigrade Thermometer, and $\alpha = \cdot 003665$.

From this we obtain

$$p = k\rho\alpha \left\{\frac{1}{\alpha} + t\right\} = K\rho T,$$

where $K = k\alpha$, and $T = \frac{1}{\alpha} + t$.

T is called the absolute temperature, the zero of which is -273° C.

In this case $\frac{dp}{p} = \frac{Xdx + Ydy + Zdz}{KT}$,

and $\therefore$ T must be a function of x, y, z.

In any of these cases, if the pressure at any particular point be given, the constant can be determined.

In the case of elastic fluids, if the mass of fluid and the space within which it is contained be given, the constant is determined.

20. The equation for determining p may also be obtained in the following manner.

Let PQ be the axis of a very small cylinder bounded by planes perpendicular to PQ.

Let p and $p + \delta p$ be the pressures at P and Q, α the areal section, and δs the length of PQ. Then, if $S\delta m$ be the component, in the direction PQ, of the forces acting on an element δm,

$$(p + \delta p)\,\alpha - p\alpha = \rho\alpha S\delta s,$$

and therefore, proceeding to the limit,

$$dp = \rho S ds.$$

That is, the rate of increase of the pressure in any direction

is equal to the product of the density and the resolved part of the force in that direction.

If x, y, z be the co-ordinates of P, and X, Y, Z the components of S parallel to the axes,

$$S = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds},$$

and $\quad \therefore dp = \rho\,(Xdx + Ydy + Zdz)$ as in Art. 16.

If the position of P be given by the cylindrical co-ordinates r, θ, and z, and if P, T, Z be the components of S in the directions of r, θ, z,

$$S = P\frac{dr}{ds} + T\frac{rd\theta}{ds} + Z\frac{dz}{ds},$$

and the equation for p becomes

$$dp = \rho\,\{Pdr + Trd\theta + Zdz\}.$$

Again, if the position of P be given by the ordinary polar co-ordinates r, θ, ϕ, and if the components of the force be R, N, and T, in the directions of r, of the perpendicular to the plane of the angle θ, and of the line perpendicular to r in that plane, it will be found that

$$\frac{dp}{\rho} = Rdr + Nr\sin\theta d\phi + Trd\theta.$$

In a similar manner the expression for dp may be obtained for any other system of co-ordinates.

21. *Surfaces of equal pressure.* In all cases, in which the equilibrium of the fluid is possible, we obtain by integration

$$p = \phi\,(x, y, z).$$

If p be constant $\quad \phi\,(x, y, z) = p$ .... (A),

is the equation to the surface at all points of which the pressure is constant, and by giving different values to p we obtain a series of surfaces of equal pressure, and the external surface, or free surface, is obtained by making p equal to the pressure external to the fluid.

If the external pressure be zero the free surface is therefore

$$\phi(x, y, z) = 0.$$

The quantities

$$\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz},$$

which are proportional to the direction-cosines of the normal at the point (x, y, z) of the surface A, are equal to

$$\frac{dp}{dx}, \frac{dp}{dy}, \frac{dp}{dz},$$

respectively, i.e. to ρX, ρY, ρZ, and are therefore proportional to X, Y, Z.

Hence the resultant force at any point is in direction of the normal to the surface of equal pressure passing through the point.

The surfaces of equal pressure are therefore the surfaces intersecting orthogonally the lines of force.

It follows from this result that a necessary condition of equilibrium is the existence of a system of surfaces orthogonal to the lines of force, a conclusion derivable also from the equation (γ) of Art. (16), for that equation is the known analytical condition requisite for the existence of such a system.

22. If the fluid be a homogeneous liquid, that is, if ρ is constant, $Xdx + Ydy + Zdz$ must be a perfect differential, or in other words, the system of forces must be a conservative system.

In general, when the force-system is conservative, ρ must be a function of the potential V.

For $dp = -\rho dV$, and dp being a perfect differential, ρ must be a function of V; hence V, and therefore ρ, is a function of p, and surfaces of equal pressure are equipotential surfaces, and are also surfaces of equal density*.

* These results may also be obtained in the following manner:

Consider two consecutive surfaces of equal pressure, containing between them a stratum of fluid, and let a small circle be described about a point P in one surface, and a portion of the fluid cut out by normals through the circumference. The portion of fluid is kept at rest by the impressed force,

If the fluid be elastic and the temperature variable

$$\frac{dp}{p} = -\frac{dV}{KT}.$$

Hence by a similar process of reasoning T is a function of p, and surfaces of equal pressure are also surfaces of equal temperature.

If however $Xdx + Ydy + Zdz$ be not a perfect differential, these surfaces will not in general coincide.

1st. Let the fluid be heterogeneous and incompressible; then the surfaces of equal pressure and of equal density are given respectively by the equations

$$dp = 0, \quad d\rho = 0,$$

or

$$\left.\begin{aligned} Xdx + Ydy + Zdz &= 0 \\ \frac{d\rho}{dx}dx + \frac{d\rho}{dy}dy + \frac{d\rho}{dz}dz &= 0 \end{aligned}\right\} \text{..............(B).}$$

These then are the differential equations of surfaces which by their intersections determine curves of equal pressure and density.

From (B) we obtain

$$\frac{dx}{Z\frac{d\rho}{dy} - Y\frac{d\rho}{dz}} = \frac{dy}{X\frac{d\rho}{dz} - Z\frac{d\rho}{dx}} = \frac{dz}{Y\frac{d\rho}{dx} - X\frac{d\rho}{dy}} \text{ .... (C).}$$

and by the pressures on its ends and on its circumference. Being very nearly a small cylinder, and the pressures at all points of its circumference being equal, the difference of the pressures on its two faces must be due to the force, which must therefore act in the same direction as these pressures, i. e. in direction of the normal at P.

If the forces are derivable from a potential, the resulting force is perpendicular to the equipotential surfaces, and the surfaces of equal pressure are therefore identical with the equipotential surfaces.

Again, considering the equilibrium of the elemental cylinder, the force acting upon it, per unit of mass, is equal to the difference of potentials divided by the distance between the surfaces of equal pressure, and as the mass of the element is directly proportional to this distance, it follows that the density must be constant, that is, the surfaces of equal pressure are also surfaces of equal density.

But from the conditions of equilibrium we have

$$\rho\frac{dX}{dy}+X\frac{d\rho}{dy}=\rho\frac{dY}{dx}+Y\frac{d\rho}{dx},$$

$$\rho\frac{dY}{dz}+Y\frac{d\rho}{dz}=\rho\frac{dZ}{dy}+Z\frac{d\rho}{dy},$$

$$\rho\frac{dZ}{dx}+Z\frac{d\rho}{dx}=\rho\frac{dX}{dz}+X\frac{d\rho}{dz},$$

and therefore the equations (C) become

$$\frac{dx}{\frac{dZ}{dy}-\frac{dY}{dz}}=\frac{dy}{\frac{dX}{dz}-\frac{dZ}{dx}}=\frac{dz}{\frac{dY}{dx}-\frac{dX}{dy}}\ \ldots\ldots\ldots\ldots \text{(D)},$$

the differential equations of the curves of equal pressure and density.

2nd. Let the fluid be elastic and of variable temperature;

$$\text{then } \frac{dp}{p}=\frac{Xdx+Ydy+Zdz}{KT},$$

and the curves of equal pressure and temperature are given by the simultaneous equations

$$dp=0, \quad dT=0;$$

$$\text{or} \quad \left.\begin{aligned} Xdx+Ydy+Zdz&=0\\ \frac{dT}{dx}dx+\frac{dT}{dy}dy+\frac{dT}{dz}dz&=0\end{aligned}\right\}.$$

But, since $\frac{dp}{p}$ is a perfect differential, the conditions of equilibrium are in this case

$$\frac{d}{dy}\cdot\frac{Z}{T}=\frac{d}{dz}\cdot\frac{Y}{T}, \text{ \&c.},$$

or

$$Z\frac{dT}{dy}-Y\frac{dT}{dz}=T\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)$$

$$X\frac{dT}{dz}-Z\frac{dT}{dx}=T\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)$$

$$Y\frac{dT}{dx}-X\frac{dT}{dy}=T\left(\frac{dY}{dz}-\frac{dX}{dy}\right).$$

But, from the preceding equations,

$$\frac{dx}{Z\frac{dT}{dy}-Y\frac{dT}{dz}}=\frac{dy}{X\frac{dT}{dz}-Z\frac{dT}{dx}}=\frac{dz}{Y\frac{dT}{dx}-X\frac{dT}{dy}};$$

$$\therefore\ \frac{dx}{\frac{dZ}{dy}-\frac{dY}{dz}}=\frac{dy}{\frac{dX}{dz}-\frac{dZ}{dx}}=\frac{dz}{\frac{dY}{dx}-\frac{dX}{dy}},$$

equations of the same form as (D), are in this case the differential equations of the curves of equal pressure and temperature, and therefore also of equal density.

23. We shall now prove that the conditions of equilibrium of a finite mass of fluid are satisfied by the equations of Art. 15.

Consider the fluid within a closed surface S, and take l, m, n as the direction-cosines of the normal at any point drawn outwards. Resolving parallel to x, and taking moments, the equations of equilibrium are

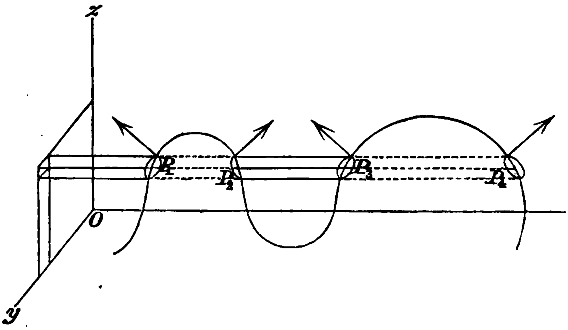
$$\iint pldS=\iiint \rho X dxdydz,$$

$$\iint pdS\,(ny-mz)=\iiint \rho\,\{Zy-Yz\}\,dxdydz.$$

Now, integrating along a thin prism parallel to x, which necessarily crosses the surface S an even number of times, and cuts out elements dS_1, dS_2, dS_3, dS_4, &c.,

$$\iiint \frac{dp}{dx}\,dxdydz=\iint pdydz,$$

between the limits P_1P_2, P_3P_4, &c.,



$=\iint -p_1 dS_1 \cos(\pi - \theta_1) + p_2 dS_2 \cos\theta_2 - p_3 dS_3 \cos(\pi - \theta_3) + \&c.,$

taking $\theta_1, \theta_2, \theta_3, \ldots$ as the inclinations to the axis of x of the outward-drawn normals,

$=\iint(p_1 l_1 dS_1 + p_2 l_2 dS_2 + \ldots) = \iint pl dS$ over the whole surface.

Transforming $\iint pny dS$ and $\iint pmz dS$ in the same manner and taking account of the equations $\frac{dp}{dx}=\rho X$, $\frac{dp}{dy}=\rho Y$, $\frac{dp}{dz}=\rho Z$, we see that the conditions of equilibrium are satisfied.

Conversely we may employ this method in order to obtain the equations alluded to.

24. We can also prove that $\rho(Xdx + Ydy + Zdz)$ must be a perfect differential, by considering the equilibrium of a spherical element of fluid.

For the pressures of the fluid on the surface of the element are all in direction of its centre, and therefore the moment of the acting forces about the centre must vanish.

Let x, y, z be co-ordinates of the centre, and $x + \alpha$, $y + \beta$, $z + \gamma$ of any point inside the small sphere.

Then, ρ being the density at the centre, the expression $\Sigma dm\,(Z\beta - Y\gamma)$ becomes

$$\iiint d\alpha d\beta d\gamma \left(\rho + \frac{d\rho}{dx}\alpha + \frac{d\rho}{dy}\beta + \frac{d\rho}{dz}\gamma\right)\left\{\beta\left(Z + \frac{dZ}{dx}\alpha + \frac{dZ}{dy}\beta + \frac{dZ}{dz}\gamma\right)\right.$$
$$\left. - \gamma\left(Y + \frac{dY}{dx}\alpha + \frac{dY}{dy}\beta + \frac{dY}{dz}\gamma\right)\right\}.$$

Now $\iiint \alpha d\alpha d\beta d\gamma = 0$, the centre of the sphere being the centre of gravity of the volume, $\iiint \beta\gamma d\alpha d\beta d\gamma = 0$, &c., and, if $\omega = d\alpha d\beta d\gamma$,

$$\iiint \alpha^2\omega = \iiint \beta^2\omega = \iiint \gamma^2\omega = \tfrac{1}{3}\iiint(\alpha^2 + \beta^2 + \gamma^2)\,\omega$$
$$= \tfrac{1}{3}\cdot\int_0^r 4\pi r'^4 dr' = \tfrac{4}{15}\pi r^5.$$

The expression for the moment then becomes, neglecting higher powers of α, β, γ,

$$\left\{\frac{d}{dy}(\rho Z) - \frac{d}{dz}(\rho Y)\right\}\frac{4\pi r^5}{15},$$

and, in order that this may be evanescent, we must have

$$\frac{d}{dy}(\rho Z) = \frac{d}{dz}(\rho Y).$$

25. *Fluid at rest under the action of gravity.*

Taking the axis of z vertical, and measuring z downwards,

$$X = 0, \quad Y = 0, \quad Z = g,$$

and the equation (α) becomes

$$dp = g\rho dz,$$

an equation which may also be obtained directly by considering the equilibrium of a small vertical cylinder.

In the case of homogeneous liquid,

$$p = g\rho z + C,$$

and the surfaces of equal pressure are horizontal planes.

Hence the free surface is a horizontal plane, and, taking the origin in the free surface, and Π as the external pressure,

$$p = g\rho z + \Pi.$$

If there be no pressure on the free surface,

$$p = g\rho z,$$

or *the pressure at any point is proportional to the depth below the surface.*

In the case of heterogeneous liquid, the equation

$$dp = g\rho dz,$$

shews that ρ must be a function of z. The density and pressure are therefore constant for all points in the same horizontal plane.

As an example, let $\rho \propto z^n = \mu z^n$,

then $$p = g\mu \frac{z^{n+1}}{n+1} + \Pi.$$

26. *If two liquids, which do not mix, meet in a bent tube, the heights of the free surfaces above the common surface are inversely as the densities.*

For the pressures at the common surface are the same, and if z, z' be the heights of the upper surfaces above the common surface, and ρ, ρ' the densities, these pressures are respectively

$$g\rho z + \Pi, \quad g\rho' z' + \Pi,$$

$$\text{and } \therefore \frac{z}{z'} = \frac{\rho'}{\rho}.$$

27. It is a well-known law that if a system be in equilibrium under the action of gravity and the pressure of smooth surfaces, the equilibrium is stable, if the centre of gravity be in its lowest possible position.

Hence it follows that, in the case of heterogeneous liquid, the density must increase with the depth, for otherwise the equilibrium would be unstable.

Thus, if heterogeneous liquid be poured from one vessel to another, it will settle with the heaviest strata lowest, the law of density of course being changed.

A quantity of liquid, the density of which is a given function of the depth, is contained in a vessel of given shape; if the liquid be transferred to another vessel, it is required to find the new law of density, each vessel being in the form of a surface of revolution with its axis vertical.

Measuring x upwards from the lowest point of the liquid, let $y = f(x)$ be the generating curve of the first vessel, and $y = \phi(x)$ of the second.

Then, if the stratum at the height x in the first vessel correspond to the stratum at the height x' in the second, we obtain, since the volumes are equal,

$$\int_0^x \{f(\xi)\}^2 d\xi = \int_0^{x'} \{\phi(\xi)\}^2 d\xi,$$

and performing the integrations, we find x in terms of x', and therefore ρ, which is a given function of x, becomes a new function of x'.

Moreover, if h and h' be the depths of the liquid in the two vessels, h is given in terms of h', and therefore the density, ρ, can be found in terms of $h'-x'$, the depth.

If the new law of density be given, and it be required to find the shape of the new vessel, we may proceed as follows:

The density being a given function of $h-x$, and also of $h'-x'$, we can, by equating the two expressions, find x in terms of x'.

Also, equating the volumes of corresponding strata, we obtain $y^2dx = y'^2dx'$, which at once, by substituting for x its value in terms of x', gives the equation required. The value of h' will be then obtained by equating to each other the whole volumes.

EXAMPLE (1). *The density of a liquid in a cylindrical vessel varies as the depth; find the new law of density if the liquid be poured into a conical vessel having its vertex downwards.*

In this case $$\rho = \mu (h-x),$$
and $$\pi a^2 x = \tfrac{1}{3}\pi x'^3 \tan^2\alpha;$$
also $$\pi a^2 h = \tfrac{1}{3}\pi h'^3 \tan^2\alpha;$$
$$\therefore \rho = \mu \tan^2\alpha \frac{h'^3 - x'^3}{3a^2} = \frac{\mu \tan^2\alpha}{3a^2}(3h'^2z - 3h'z^2 + z^3),$$
if z be the depth.

EXAMPLE (2). *A quantity of liquid the density of which varies as the depth, fills an inverted paraboloid to a given height; it is required to find the shape of a vessel, in the form of a surface of revolution, such that if this liquid be poured into it its density will vary as the square of its depth.*

In this case $\rho = \mu (h-x) = \mu' (h'-x')^2$,
$$\therefore x = h - \frac{1}{c}(h'-x')^2, \text{ if } \mu = \mu' c.$$

The equation $4axdx = y'^2dx'$ gives
$$c^2y'^2 = 8a(h'-x')\{hc - (h'-x')^2\}.$$

To complete the solution, we must equate the total

volumes, and we thereby obtain $h'^2 = ch$ as the necessary relation between h' and c.

28. *Elastic fluid at rest under the action of gravity.*

In this case, $$p = k\rho,$$

and $$\frac{dp}{p} = \frac{g}{k}\,dz;$$

$$\therefore \log\frac{p}{C} = \frac{gz}{k} \text{ and } p = C\epsilon^{\frac{gz}{k}}.$$

The surfaces of equal pressure are in this case also horizontal planes, and the constant C must be determined by a knowledge of the pressure for a given value of z, or by some other fact in connection with the particular case.

EXAMPLE. *A closed cylinder, the axis of which is vertical, contains a given mass of air.*

Measuring z from the top of the cylinder,

$$\rho = \frac{p}{k} = \frac{C}{k}\,\epsilon^{\frac{gz}{k}},$$

$\therefore$ if M be the given mass, a the radius, and h the height of the cylinder,

$$M = \int_0^h \rho\pi a^2 dz = \pi a^2 \frac{C}{g}\left(\epsilon^{\frac{gh}{k}} - 1\right),$$

whence C is determined.

29. *Illustrations of the use of the general equation.*

(1) Let a given volume V of liquid be acted upon by forces

$$-\frac{\mu x}{a^2},\ -\frac{\mu y}{b^2},\ -\frac{\mu z}{c^2},$$

respectively parallel to the axes;

then $$dp = \rho\left(-\frac{\mu x}{a^2}\,dx - \frac{\mu y}{b^2}\,dy - \frac{\mu z}{c^2}\,dz\right),$$

and $$p = C - \frac{\mu\rho}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right).$$

The surfaces of equal pressure are therefore similar ellipsoids, and the equation to the free surface is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{2C}{\mu\rho},$$

assuming that there is no external pressure.

The condition which determines the constant is that the volume of the fluid is given, and we have

$$V=\tfrac{4}{3}\pi abc\,.\left(\frac{2C}{\mu\rho}\right)^{\frac{3}{2}},$$

$$\text{and } C=\frac{\mu\rho}{2}\,.\left(\frac{3V}{4\pi abc}\right)^{\frac{2}{3}}.$$

(2) *A given volume of liquid is at rest on a fixed plane, under the action of a force, to a fixed point in the plane, varying as the distance.*

Taking the fixed point as origin, the expression for the pressure at any point is

$$p=C-\tfrac{1}{2}\mu\rho\,(x^2+y^2+z^2)=C-\tfrac{1}{2}\mu\rho r^2,$$

where r is the distance from the origin; and if $\frac{2}{3}\pi a^3$ be the given volume, the free surface is a hemisphere of radius a, and

$$p=\tfrac{1}{2}\mu\rho\,(a^2-r^2).$$

The portion of the plane in contact with fluid is a circle of radius a, and therefore the pressure upon it

$$=\int_0^{2\pi}\int_0^a prdrd\theta$$

$$=\tfrac{1}{4}\pi\mu\rho a^4.$$

This result may be written in the form $\mu\frac{3}{8}a\,.\,\frac{2}{3}\pi\rho a^3$, which is the expression for the attraction on the whole mass of fluid, supposed to be condensed into a material particle at its centre of gravity, and might in fact have been at once obtained by considering that the fluid is kept at rest by the attraction to the centre of force and the reaction of the plane.

(3) *A given volume of heavy liquid is at rest under the action of a force to a fixed point varying as the distance from that point.*

Take the fixed point as origin, and measure z vertically downwards;

then $$X = -\mu x,\quad Y = -\mu y, \text{ and } Z = g - \mu z;$$

$$\therefore dp = \rho\,\{-\mu x dx - \mu y dy + (g - \mu z)\,dz\},$$

and $$\frac{p}{\rho} = C - \mu\frac{x^2 + y^2 + z^2}{2} + gz.$$

The surfaces of equal pressure are spheres, and the free surface supposing the external pressure zero, is given by the equation

$$x^2 + y^2 + z^2 - \frac{2g}{\mu}z = \frac{2C}{\mu}.$$

The volume of this sphere is

$$\tfrac{4}{3}\pi\left(\frac{2C}{\mu} + \frac{g^2}{\mu^2}\right)^{\frac{3}{2}};$$

equating this to the given volume, the constant C is determined, and the pressure at any point is then given in terms of r and z.

Rotating Fluid.

30. If a quantity of fluid revolve uniformly and without any relative displacement of its particles (i.e. as if rigid) about a fixed axis, the preceding equations will enable us to determine the pressure at any point, and the nature of the surfaces of equal pressure.

For, in such cases of relative equilibrium, every particle of the fluid moves uniformly in a circle, and the resultant of the external forces acting on any particle m of the fluid, and of the fluid pressure upon it, must be equal to a force $m\omega^2 r$ *towards* the axis, ω being the angular velocity, and r the distance of m from the axis; it follows therefore that the external forces, combined with the fluid pressures and forces $m\omega^2 r$ acting *from* the axis, form a system in statical equilibrium, to which the equations of the previous articles are applicable.

A mass of homogeneous liquid, contained in a vessel, revolves uniformly about a vertical axis; required to deter-

mine the pressure at any point, and the surfaces of equal pressure.

Take the vertical axis as the axis of z; then, resolving the force $m\omega^2 r$ parallel to the axes, its components are $m\omega^2 x$ and $m\omega^2 y$, and the general equation of fluid equilibrium becomes

$$dp = \rho\,(\omega^2 x dx + \omega^2 y dy - g dz),$$

and therefore

$$p = \rho\,\{\tfrac{1}{2}\omega^2 (x^2 + y^2) - gz\} + C.$$

The surfaces of equal pressure are therefore paraboloids of revolution, and if the vessel be open at the top, the free surface is given by the equation

$$\omega^2 (x^2 + y^2) - 2gz + \frac{2C}{\rho} = \frac{2\Pi}{\rho},$$

where Π is the external pressure.

The constant must be determined by help of the data of each particular case.

For instance, let the vessel be closed at the top and be just filled with liquid, and let $\Pi = 0$; then, taking the origin at the highest point of the axis, $p = 0$ when x, y and z vanish, and therefore $C = 0$, and

$$p = \rho\,\{\tfrac{1}{2}\omega^2 (x^2 + y^2) - gz\}.$$

Next consider the case of elastic fluid enclosed in a vessel which rotates about a vertical axis;

as before $$dp = \rho\,\{\omega^2 (x dx + y dy) - g dz\};$$

$$\text{and} \quad p = k\rho;$$

$$\therefore\ k \log \rho = \omega^2 \frac{x^2 + y^2}{2} - gz + C,$$

so that the surfaces of equal pressure and density are paraboloids.

Let the containing vessel be a cylinder rotating about its axis, and suppose the whole mass of fluid given; then, to determine the constant, consider the fluid arranged in ele-

mentary horizontal rings each of uniform density: let r be the radius of one of these rings at a height z, δr its horizontal and δz its vertical thickness, h the height, and a the radius of the cylinder:

$$\text{the mass of the ring} = 2\pi\rho r\delta r\delta z,$$

$$\text{and the whole mass } (M) \text{ of the fluid} = \int_0^h \int_0^a 2\pi\rho r dr dz,$$

the origin being taken at the base of the cylinder.

$$\text{Now } \rho = \epsilon^{\frac{C}{k}} \cdot \epsilon^{\frac{\omega^2 r^2 - 2gz}{2k}};$$

$$\text{and } \therefore M = \frac{2\pi k^2}{g\omega^2} \epsilon^{\frac{C}{k}} (\epsilon^{\frac{\omega^2 a^2}{2k}} - 1)(1 - \epsilon^{-\frac{gh}{k}}),$$

an equation by which C is determined.

31. In general the equation of equilibrium for a fluid revolving uniformly and acted upon by forces of any kind, is

$$dp = \rho \{Xdx + Ydy + Zdz + \omega^2 (xdx + ydy)\}.$$

In order that the equilibrium may be possible, three equations of condition must be satisfied, expressing that dp is a perfect differential, and, if these conditions are satisfied, the surfaces of equal pressure, and, in certain cases, the free surface can be determined; but it must be observed that a free surface is not always possible. In fact, in order that there may be a free surface, the surfaces of equal pressure must be symmetrical with respect to the axis of rotation.

Whole Pressure.

32. DEF. *The whole pressure of a fluid on any surface with which it is in contact is the sum of the normal pressures on each of its elements.*

If then p be the pressure at a point of an element δS of the surface,

$$p\delta S \text{ is the pressure on the element,}$$

and $\iint p dS$ is the whole pressure, the summation extending over the whole of the surface considered.

If the fluid be homogeneous liquid, and gravity the only force in action, $p = g\rho z$, measuring z vertically downwards from the surface of the liquid,

$$\text{and } \iint p dS = \iint g\rho z S.$$

Let $\bar{z}$ be the depth of the centre of gravity of the *surface S*,

$$\text{then } \bar{z} . S = \iint z dS;$$

$$\text{and } \therefore \text{ the whole pressure} = g\rho\bar{z}S,$$

i.e. the whole pressure is equal to the weight of a cylindrical column of fluid, the height of which is $\bar{z}$, and the base a plane area equal to the area of the surface.

We now add some examples of the determination of whole pressure.

(1) *A hemispherical bowl filled with water.*

Let r be its radius, ρ the density of water.

Then the surface $= 2\pi r^2$,

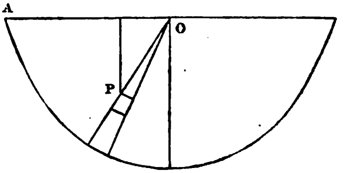
$$\text{and } \bar{z} = \frac{r}{2};$$

$$\therefore \text{ whole pressure} = g\rho\pi r^3,$$

i.e. whole pressure : the weight of the fluid :: 3 : 2.

(2) *The density of a heavy liquid varies as the square of the depth; it is required to find the whole pressure on a semicircular area immersed vertically with its bounding diameter in the surface.*

Let $OP = r$, $AOP = \theta$;



then (Art. 25) if the density $= \mu$ (depth)^2, the pressure at P

$$= \frac{\mu g}{3} (r \sin \theta)^3,$$

and the whole pressure

$$= 2 \int_0^a \int_0^{\frac{\pi}{2}} \frac{\mu g}{3} r^4 \sin^3 \theta d\theta dr,$$

$$= \tfrac{4}{45} \mu g a^5.$$

(3) *A cylindrical vessel is closed at the top, and very nearly filled with incompressible fluid, which rotates uniformly about the axis of the cylinder; to find the whole pressure on the curved surface and on the top of the cylinder.*

In this case, taking the centre of the top as origin, and measuring z downwards,

$$\frac{p}{\rho}=\frac{\omega^2 r^2}{2}+gz.$$

Let a be the radius of the cylinder, h its height; then at a depth z, the pressure at its surface

$$=\rho\left(\frac{\omega^2 a^2}{2}+gz\right),$$

an element of surface $=2\pi a\,.\,\delta z$;

$\therefore$ the whole pressure on the curved surface

$$=\int_0^h 2\pi a\rho\left(\tfrac{1}{2}\omega^2 a^2+gz\right)dz,$$

$$=\pi\rho a^3 h\omega^2+\pi\rho a g h^2.$$

The pressure on the top at a distance r from the origin $=\frac{1}{2}\rho\omega^2 r^2$,

and an element of its area $=2\pi r\delta r$;

therefore the whole pressure on the top

$$=\int_0^a \pi\rho\omega^2 r^3\,dr=\tfrac{1}{4}\pi\rho\omega^2 a^4.$$

(4) *A hollow spherical shell is just filled with homogeneous liquid, and the liquid is at rest under the action of a force, to a point on the inner surface of the shell, proportional to the distance from that point; it is required to find the whole pressure on the shell.*

Let O be the centre of force, and r the distance of any point from O.

Then $$dp=-\mu\rho r dr,$$

and $$p=C-\mu\rho\,\frac{r^2}{2}.$$

The pressure vanishes at the other extremity of the diameter OA, and therefore

$$p = \mu\rho\left(2a^2 - \frac{r^2}{2}\right),$$

a being the radius AC.

If P be a point in the sphere and $ACP = \theta$,

$$\text{then } OP = 2a\cos\frac{\theta}{2},$$

and the pressure at $P = 2\mu\rho a^2 \sin^2\frac{\theta}{2}$.

If $PCQ = \delta\theta$, in the plane of θ, the surface generated by the revolution of the arc PQ about OA

$$= 2\pi a\delta\theta \,.\, a\sin\theta,$$

and $\therefore$ the whole pressure on the surface

$$= \int_0^\pi 4\pi\mu\rho a^4 \sin^2\frac{\theta}{2}\sin\theta d\theta$$

$$= 2\pi\mu\rho a^4 \int_0^\pi (1 - \cos\theta)\sin\theta d\theta$$

$$= 4\pi\mu\rho a^4.$$

EXAMPLES.

1. A closed tube in the form of an ellipse with its major axis vertical is filled with three different liquids of densities ρ_1, ρ_2, ρ_3 respectively. If the distances of the surfaces of separation from either focus be r_1, r_2, r_3 respectively, prove that

$$r_1(\rho_2 - \rho_3) + r_2(\rho_3 - \rho_1) + r_3(\rho_1 - \rho_2) = 0.$$

2. A heavy uniform fluid just fills a sphere: shew that a plane drawn through a horizontal tangent at an extremity of a horizontal diameter will divide the surface of the sphere into parts the whole pressures on which are equal, if it is inclined at an angle $\tan^{-1}x$ to the vertical where $x^3 - 2x^2 - 2 = 0$.

3. The particles of a given mass of homogeneous liquid at rest attract each other according to the law of nature; find the pressure at any point.

4. The density of a liquid varies as the square of the depth below the surface; find the whole pressures, 1st, on a rectangular area just immersed vertically with one side in the surface, 2nd, on a circular area just immersed.

5. A parabolic area, bounded by the latus rectum, is just immersed vertically, with its vertex in the surface of a liquid; find the whole pressure upon it, 1st, when the liquid is homogeneous, 2nd, when its density varies as the depth.

6. Find the surfaces of equal pressure when the forces tend to fixed centres and vary as the distances from those centres.

7. A regular tetrahedron is filled with fluid, and held so that two of its opposite edges are horizontal; compare the pressures on its several sides with the weight of the fluid.

8. A spherical mass of elastic fluid is compressed into the cube which can be inscribed within the sphere; compare the whole pressures on the surfaces of the cube and sphere.

If a mass of air in a cubical vessel be compressed into the sphere which can be inscribed in the cube, the whole pressures on the two surfaces are equal.

9. In a solid sphere two spherical cavities, whose radii are equal to half the radius of the solid sphere, are filled with liquid; the solid and liquid particles attract each other with forces which vary as the distance: prove that the surfaces of equal pressure are spheres concentric with the solid sphere.

10. Shew that the forces represented by

$$X = \mu\,(y^2 + yz + z^2),\quad Y = \mu\,(z^2 + zx + x^2),\quad Z = \mu\,(x^2 + xy + y^2)$$

will keep a mass of liquid at rest, if the density $\propto \dfrac{1}{(\text{dist.})^2}$ from the plane $x + y + z = 0$; and the curves of equal pressure and density will be circles.

11. A given quantity of elastic fluid is contained in a hollow sphere, and its particles are acted upon by a force to

the centre of the sphere varying inversely as the distance. The sphere being supposed to vary in size, shew that the whole pressure on its surface varies inversely as its radius, provided $\mu < 3\kappa$, where μ is the absolute force, and κ the ratio of the pressure to the density of the fluid.

12. A closed cylindrical vessel is very nearly filled with incompressible fluid, which is acted upon by a force, varying as the distance, to the middle point of the axis of the cylinder; if $2a$ be the length of the axis and c the radius of either end, shew that the whole pressure on the curved surface : the whole pressure on the ends :: $8a^3 : 3c^3$.

Also find this ratio when the centre of force is at the centre of either end of the cylinder.

13. If a conical cup be filled with liquid, the mean pressure at a point in the volume of the liquid is to the mean pressure at a point in the surface of the cup as 3 : 4.

14. A vessel is in the form of a right cone without weight, the vertical angle being $2a$; the vessel is filled with liquid and then suspended by a point in the rim: if β be the inclination of the axis of the cone to the vertical, shew that

$$\cot 2\beta = \cot 2a - \frac{3}{4} \operatorname{cosec} 2a.$$

15. A mass of fluid rests upon a plane subject to a central attractive force $\left(\frac{\mu}{r^2}\right)$, situated at a distance c from the plane on the side opposite to that on which the fluid is; and a is the radius of the free spherical surface of the fluid: shew that the whole pressure on the plane

$$= \frac{\pi\rho\mu}{a} (a - c)^2.$$

16. Find the surfaces of equal pressure for fluid acted upon by two forces which vary as the inverse square of the distance from two fixed points.

Prove that if the surface of no pressure be a sphere, the loci of points at which the pressure varies inversely as the distance from one of the centres of force are also spheres.

17. The unit of velocity being a velocity of one foot per second, and the unit of acceleration that of a falling body, find the gravitation unit of force in the equation $p = g\rho z$, water being taken as the standard substance.

18. A cylindrical rod, of radius one inch and length eight inches, is placed in a vessel of water ten inches deep, with one end on the bottom of the vessel, and is inclined to the vertical at an angle $45°$; if an inch be the unit of length, a yard per second the unit of velocity, and the density of water the unit of density, find the number of gravitation units of force in the whole pressure on the rod.

19. A closed cylinder, with its axis vertical, is just filled with liquid which rotates uniformly about a generating line; find the whole pressures on the base, the upper end, and the curved surface.

20. A vessel in the form of an inverted cone is partly filled with fluid, and closed with a lid; it is then made to revolve uniformly about its axis; if a small hole be now made at the vertex, determine how much of the fluid will escape, considering the different cases that arise according to the magnitude of the angular velocity. If this be indefinitely increased, prove that the surface of the fluid is a circular cylinder, and find its radius.

21. If the force at any point is given by a potential ϕ, and if a tube of small but variable circular section be imagined in the liquid, the whole pressure upon which is P, prove that

$$\frac{d^2P}{d\phi ds} + 2\pi\rho r = 0$$

where r is the radius of the section, and s is measured along the axis of the tube.

22. The density of a liquid, contained in a cylindrical vessel, varies as the depth; it is transferred to another vessel, in which the density varies as the square of the depth; find the shape of the new vessel.

23. A circular cone, of vertical angle $\frac{\pi}{3}$, is just filled with water, and has a generating line rigidly attached to a horizontal plane. The plane is caused to revolve with uniform angular velocity about a vertical axis through the apex of the cone: find the greatest velocity which will allow of the pressure being zero at the highest point; and in this case find the whole pressure on the base.

24. A straight rod, every particle of which attracts with a force varying inversely as the square of the distance, is surrounded by a mass of homogeneous incompressible fluid; find the form of the surfaces of equal pressure.

25. A quantity of heavy liquid is attracted to a fixed centre, by a constant force the intensity of which is equal to the force of gravity, and is supported by a horizontal plane. Find the form of the surfaces of equal pressure; and also the pressure on the plane, proving that when the plane passes through the centre of force it is equal to four-thirds of the weight of the liquid. Find also expressions for the pressure on the plane when it is either above or below the centre of force.

26. The interior of a homogeneous shell, bounded by two non-concentric spherical surfaces, and attracting according to the law of nature, is partially filled with homogeneous liquid which revolves uniformly with it round the line passing through the centres of the spheres; prove that the free surface is a paraboloid of revolution.

27. A rigid spherical shell is filled with homogeneous inelastic fluid, every particle of which attracts every other with a force varying inversely as the square of the distance; shew that the difference between the pressures at the surface and at any point within the fluid varies as the area of the least section of the sphere through the point.

28. At the vertex of a solid cone (vertical angle 2α) there is a centre of force the attraction to which varies as the distance; and a given quantity of liquid is in equilibrium under the action of this force alone. Determine the form of its

free surface. If the volume of the liquid be $\frac{4}{3}\pi a^3 \cos^2\frac{\alpha}{2}$, prove that the whole pressure on the surface of the cone $=\frac{1}{4}\mu\rho\pi a^4 \sin\alpha$, where ρ is the density of the liquid and μ the absolute force.

29. An open vessel containing liquid is made to revolve about a vertical axis with uniform angular velocity. Find the form of the vessel and its dimensions in order that it may be just emptied.

30. A quantity of liquid (gravity being supposed not to act) just fills a hollow sphere, and is repelled from a point in the surface of the sphere by a force $=\mu\times$ distance: if the liquid revolve round the diameter passing through the centre of force with uniform angular velocity ω, find the whole pressure on the surface of the sphere. If, by diminishing the angular velocity one half, the pressure is also diminished one half, shew that $\omega^2 = 6\mu$.

31. A rectangular plate of thin metal of given size is bent and held so that two opposite edges are parallel and in the same horizontal plane, and the vertical ends are then closed by flat plates; if this vessel be filled with water, find its form when the whole pressure upon its curved surface is a maximum.

32. An infinite mass of homogeneous fluid surrounds a closed surface and is attracted to a point (O) within the surface with a face which varies inversely as the cube of the distance. If the pressure on any element of the surface about a point P be resolved along PO, prove that the whole radial pressure, thus estimated, is constant, whatever be the shape and size of the surface, it being given that the pressure of the fluid vanishes at an infinite distance from the point O.

33. A right cone, whose weight may be neglected, is suspended from a point in its rim; it contains as much fluid as it can: prove that the whole pressure upon its surface is

$$\frac{1}{3}\pi\rho g h^3 \frac{\sin\alpha\cos\theta}{\cos^2\alpha}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}},$$

where h, 2α, are the height and vertical angle of the cone, and θ is determined from $3 \sin 2\theta = 4 \sin 2(\theta - \alpha)$.

34. A vessel formed by the revolution of a cardioid $r = a(1 - \cos\theta)$ about its axis which is vertical (vertex upwards) is just filled with water and rotates about that axis with uniform angular velocity. Find this velocity, when the line of no pressure is given by $\theta = \frac{\pi}{6}$. Find also the pressure at any other point, and the points of maximum pressure.

35. A closed vessel full of liquid is made to revolve with uniform angular velocity ω about a vertical axis through its highest point; shew that the total pressure of the liquid on the surface is increased by $\frac{1}{2} Ak^2\rho\omega^2$; A being the area of the surface, and k the radius of gyration of the surface about the vertical axis.

36. All space being supposed filled with an elastic fluid the particles of which are attracted to a given point by a force varying as the distance, and the whole mass of the fluid being given, find the pressure on a circular disc placed with its centre at the centre of force.

37. A thin ellipsoidal shell, attracting according to the law of nature, is surrounded by homogeneous liquid; find the surfaces of equal pressure, neglecting the attraction of the liquid on itself.

38. Circles are drawn having their centres on the axis of z and touching at the origin the plane xy, and the position of a point P is defined by r, θ, ϕ, where r is the radius of the circle through P, centre C, θ is the angle OCP, and ϕ the inclination of the plane OCP to a fixed plane through the axis of z; prove that

$$\frac{dp}{\rho} = R(1 - \cos\theta)\, dr + T \sin\theta\, dr + Tr d\theta + Nr \sin\theta\, d\phi,$$

where mR, mT, mN are the forces, on an element m of liquid at P, along CP, along the tangent to the circle at P, and perpendicular to the plane of the circle.

39. A mass m of elastic fluid is rotating about an axis with uniform angular velocity ω, and is acted on by an attraction towards a point in that axis equal to μ times the distance, μ being greater than ω^2; prove that the equation of a surface of equal density ρ is

$$\mu(x^2+y^2+z^2)-\omega^2(x^2+y^2)=k\log\left\{\frac{\mu(\mu-\omega^2)^2}{8\pi^3}\cdot\frac{m^2}{\rho^2k^3}\right\}.$$

40. A quantity of liquid, the density of which varies as the depth, fills an inverted paraboloid, of latus rectum c, to a height h; prove that, if it be poured into a vessel of the form generated by the revolution round the axis of x of the curve,

$$a^4y^2=2ch^2x(a-x)(2a-x),$$

where a is any constant, its density will vary as the square of its depth.

41. A mass of self-attracting liquid, of density ρ, is in equilibrium, the law of attraction being that of the inverse square: prove that the mean pressure throughout any sphere of the liquid, of radius r, is less by $\frac{2}{5}\pi\rho^2r^2$ than the pressure at its centre.

CHAPTER III.

THE RESULTANT PRESSURE OF FLUIDS ON SURFACES.

33. In the preceding Chapter we have shewn how to investigate the pressure *at any point* of a fluid at rest under the action of given forces; we now proceed to determine the resultants of the pressures exerted by fluids *upon surfaces* with which they are in contact.

We shall consider, first, the action of fluids on plane surfaces, secondly, of fluids under the action of gravity upon curved surfaces, and thirdly, of fluids at rest under any given forces upon curved surfaces.

Fluid Pressures on Plane Surfaces.

The pressures at all points of a plane being perpendicular to it, and in the same direction, the resultant pressure is equal to the sum of these pressures, that is, to the whole pressure, and acts in the same direction.

Hence, if the fluid be incompressible and acted upon by gravity only, the resultant pressure on a plane

$$= \text{the whole pressure}$$

$$= g\rho\bar{z}A,$$

where A is the area and $\bar{z}$ the depth of the centre of gravity.

In general, if the fluid be of any kind, and at rest under the action of any given forces, take the axes of x and y in plane, and let p be the pressure at the point (x, y).

The pressure on an element of area $\delta x \delta y = p\delta x \delta y$;

$$\therefore \text{ the resultant pressure} = \iint p\,dy\,dx,$$

the integration extending over the whole of the area considered.

If polar co-ordinates be used, the resultant pressure is given by the expression

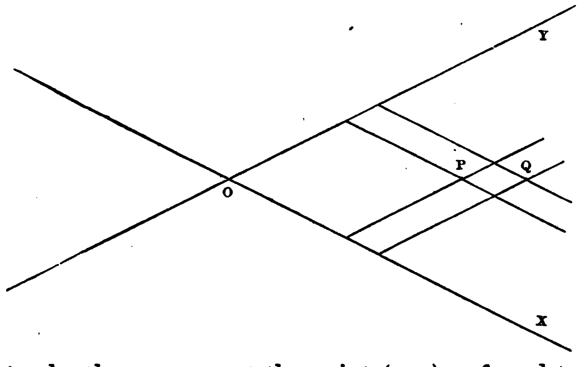
$$\iint pr\,dr\,d\theta.$$

34. Def. *The centre of pressure is the point at which the direction of the single force, which is equivalent to the fluid pressures on the plane surface, meets the surface.*

The centre of pressure is here defined with respect to plane surfaces only; it will be seen afterwards that the resultant action of fluid on a curved surface is not always reducible to a single force.

In the case of a heavy fluid, it is clear that the centre of pressure of a horizontal area, the pressure on every point of which is the same, is its centre of gravity; and, since pressure increases with the depth, the centre of pressure of any plane area, not horizontal, is below its centre of gravity.

To obtain formulæ for the determination of the centre of pressure of any plane area.



Let p be the pressure at the point (x, y), referred to rectangular axes in the plane, $x + \delta x$, $y + \delta y$, the co-ordinates of Q,

$\bar{x}$, $\bar{y}$, co-ordinates of the centre of pressure.

Then $\bar{y} \,.\, \iint p\,dy\,dx$ = moment of the resultant pressure about OX,

= the sum of the moments of the pressures on all the elements of area about OX,

$= \Sigma\, p\delta y\, \delta x \,.\, y$

$= \iint py\,dy\,dx$;

$$\therefore\ \bar{y} = \frac{\iint py\,dy\,dx}{\iint p\,dy\,dx},$$

and similarly
$$\bar{x} = \frac{\iint px\,dy\,dx}{\iint p\,dy\,dx},$$

the integrals being taken so as to include the area considered.

If polar co-ordinates be employed, a similar process will give the equations

$$\bar{x} = \frac{\iint pr^2 \cos\theta\, dr\, d\theta}{\iint pr\, dr\, d\theta}, \quad \bar{y} = \frac{\iint pr^2 \sin\theta\, dr\, d\theta}{\iint pr\, dr\, d\theta}.$$

35. If the fluid be homogeneous and inelastic, and if gravity be the only force in action,

$$p = g\rho h,$$

where h is the depth of the point P below the surface; and we obtain

$$\bar{x} = \frac{\iint hx\,dy\,dx}{\iint h\,dy\,dx}, \quad \bar{y} = \frac{\iint hy\,dy\,dx}{\iint h\,dy\,dx} \quad \ldots\ldots\ldots (\alpha).$$

It is sometimes useful to take for one of the axes the line of intersection of the plane with the surface of the fluid: if we take this line for the axis of x, and θ as the inclination of the plane to the horizon, $p = g\rho y \sin\theta$, and therefore

$$\bar{x} = \frac{\iint xy\,dy\,dx}{\iint y\,dy\,dx}, \quad \bar{y} = \frac{\iint y^2\,dy\,dx}{\iint y\,dy\,dx} \ldots\ldots\ldots (\beta).$$

From these last equations (β) it appears that the position of the centre of pressure is independent of the inclination of the plane to the horizon, so that if a plane area be immersed in fluid, and then turned about its line of intersection with

the surface as a fixed axis, the centre of pressure will remain unchanged.

If in the equations (α) we make h constant, that is, if we suppose the plane horizontal, $\bar{x}$ and $\bar{y}$ are the co-ordinates of the centre of gravity of the area, a result in accordance with Art. (34); but, in the equations (β), the values of $\bar{x}$ and $\bar{y}$ are independent of θ, and are therefore unaffected by the evanescence of θ. This apparent anomaly is explained by considering that, however small θ be taken, the portion of fluid between the plane area and the surface of the fluid is always wedge-like in form, and the pressures at the different points of the plane, although they all vanish in the limit, do not vanish in ratios of equality, but in the constant ratios which they bear to one another for any finite value of θ.

The equations of this article may also be obtained by the following reasoning.

Through the boundary line of the plane area draw vertical lines to the surface enclosing a mass of fluid; then the reaction of the plane, resolved vertically, is equal to the weight of the fluid, which acts in a vertical line through its centre of gravity; and the point in which this line meets the plane is the centre of pressure.

Taking the same axes, the weight of an elementary prism, acting through the point x, y, is $g\rho h\delta x\delta y \cos\theta$, where θ is the inclination of the plane to the horizon; and therefore the centre of these parallel forces acting at points of the plane, is given by the equations

$$\bar{x} = \frac{\iint g\rho h\, x \cos\theta\, dy\, dx}{\iint g\rho h \cos\theta\, dy\, dx}, \quad \bar{y} = \frac{\iint g\rho h\, y \cos\theta\, dy\, dx}{\iint g\rho h \cos\theta\, dy\, dx},$$

or $$\bar{x} = \frac{\iint hx\, dy\, dx}{\iint h\, dy\, dx}, \quad \bar{y} = \frac{\iint hy\, dy\, dx}{\iint h\, dy\, dx}.$$

Hence it appears that the depth of the centre of pressure is double that of the centre of gravity of the fluid enclosed.

36. The following theorem determines geometrically the position of the centre of pressure for the case of a heavy liquid.

If a straight line be taken in the plane of the area, parallel to the surface of the liquid and as far below the centre of inertia of the area as the surface of the liquid is above, the pole of this straight line with respect to the momental ellipse at the centre of inertia whose semi-axes are equal to the principal radii of gyration at that point will be the centre of pressure of the area.

Taking A for the area, and b, a for the principal radii of gyration, these are determined by the equations

$$Ab^2 = \iint y^2\, dxdy, \quad Aa^2 = \iint x^2\, dxdy,$$

and the equation of the momental ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the co-ordinate axes being the principal axes at the centre of inertia.

Let $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre of pressure, and

$$x \cos\theta + y \sin\theta = p$$

the equation to the line in the surface;

then $$\bar{x} = \frac{\iint (p - x\cos\theta - y\sin\theta)\, xdxdy}{\iint (p - x\cos\theta - y\sin\theta)\, dxdy} = -\frac{a^2}{p}\cos\theta,$$

and similarly, $$\bar{y} = -\frac{b^2}{p}\sin\theta;$$

$\therefore$ $(\bar{x}, \bar{y})$ is the pole of the line

$$x\cos\theta + y\sin\theta = -p$$

with respect to the momental ellipse.

37. *Examples of the determination of centres of pressure.*

(1) *A quadrant of a circle just immersed vertically in a heavy homogeneous liquid, with one edge in the surface.*

If Ox, the edge in the surface, be the axis of x,

$$\bar{x} = \frac{\int_0^a \int_0^{\sqrt{(a^2-x^2)}} xy\, dx\, dy}{\int_0^a \int_0^{\sqrt{(a^2-x^2)}} ydxdy}, \quad \bar{y} = \frac{\iint y^2\, dxdy}{\iint ydxdy},$$

the limits of the integrations for $\bar{y}$ being the same as for $\bar{x}$.

$$\iint y\,dx\,dy = \tfrac{1}{2}\int (a^2 - x^2)\,dx = \tfrac{1}{3}a^3,$$

$$\iint xy\,dx\,dy = \tfrac{1}{2}\int x\,.\,(a^2 - x^2)\,dx = \tfrac{1}{8}a^4,$$

$$\iint y^2\,dx\,dy = \tfrac{1}{3}\int (a^2 - x^2)^{\frac{3}{2}}\,dx = \frac{\pi a^4}{16};$$

$$\therefore\ \bar{x} = \frac{3}{8}a,\quad \bar{y} = \frac{3}{16}\pi a.$$

Employing polar co-ordinates and taking the line Ox as the initial line, we should have $p = g\rho r \sin\theta$, and

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}}\int_0^a r^3 \cos\theta \sin\theta\, dr\, d\theta}{\iint r^2 \sin\theta\, dr\, d\theta} = \frac{3}{8}a,$$

$$\text{and } \bar{y} = \frac{\int_0^{\frac{\pi}{2}}\int_0^a r^3 \sin^2\theta\, dr\, d\theta}{\iint r^2 \sin\theta\, dr\, d\theta} = \frac{3}{16}\pi a.$$

(2) *A circular area, radius* a, *is immersed with its plane vertical, and its centre at a depth* c.

Take the centre as the origin, and the vertical downwards from the centre as the initial line; then if p be the pressure at the point r, θ,

$$p = g\rho\,(c + r\cos\theta),$$

and the depth below the centre of the centre of pressure

$$= \frac{2\int_0^{\pi}\int_0^a r^2 \cos\theta\,(c + r\cos\theta)\, dr\, d\theta}{2\iint r\,(c + r\cos\theta)\, dr\, d\theta} = \frac{a^2}{4c}.$$

It will be seen that this result is at once given by the theorem of Art. 36.

(3) *A vertical rectangle, exposed to the action of the atmosphere at a constant temperature.*

If Π be the atmospheric pressure at the base of the rectangle, the pressure at a height z is $\Pi\epsilon^{-\frac{gz}{k}}$, Art. (28), and if b denote the breadth, the pressure upon a horizontal strip of the rectangle

$$= \Pi\epsilon^{-\frac{gz}{k}} b\delta z,$$

$\therefore$ the resultant pressure, if a be the height,

$$= \int_0^a \Pi\epsilon^{-\frac{gz}{k}} bdz = \Pi \frac{bk}{g}(1 - \epsilon^{-\frac{ga}{k}}),$$

and the height of the centre of pressure

$$= \frac{\int_0^a ze^{-\frac{gz}{k}} dz}{\int_0^a \epsilon^{-\frac{gz}{k}} dz} = \frac{k}{g} - \frac{a}{\epsilon^{\frac{ga}{k}} - 1}.$$

(4) *A hollow cube is very nearly filled with liquid, and rotates uniformly about a diagonal which is vertical; required to find the pressures upon, and the centres of pressure of, its several faces.*

I. For one of the upper faces $ABCD$,

take AD, AB, as axes of x and y; z, r, the vertical and horizontal distances of any point $P(x, y)$ from A,

then $\frac{p}{\rho} = \frac{1}{2}\omega^2 r^2 + gz,$

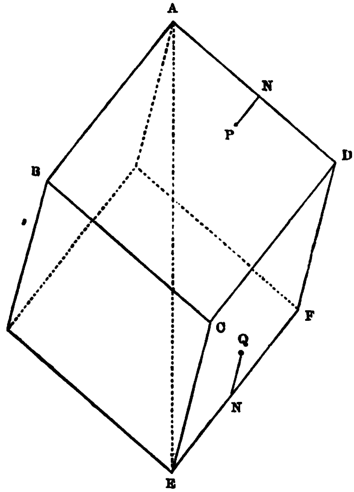
$$z = \frac{x+y}{\sqrt{3}}, \text{ projecting the broken line } ANP \text{ on } AE,$$

$$r^2 = AP^2 - z^2 = x^2 + y^2 - z^2 = \tfrac{2}{3}(x^2 + y^2 - xy);$$

$\therefore$ the pressure (P) on $ABCD = \int_0^a\!\!\int pdy\,dx$

$$= \rho \cdot \iint \left\{\frac{\omega^2}{3}(x^2 + y^2 - xy) + \frac{g}{\sqrt{3}}(x+y)\right\} dydx$$

$$= \rho \left\{\frac{5}{36} a^4\omega^2 + \frac{g}{\sqrt{3}} a^3\right\}.$$



The centre of pressure is given by the equations

$$\bar{x}P = \bar{y}P = \rho\int_0^a\int_0^a x\left\{\frac{\omega^2}{3}(x^2+y^2-xy)+\frac{g}{\sqrt{3}}(x+y)\right\}dy\,dx\,;$$

$$\therefore\ \bar{x} = \bar{y} = a\,.\,\frac{21g+3\sqrt{3}\omega^2 a}{36g+5\sqrt{3}\omega^2 a}\,.$$

II. For one of the lower faces $ECDF$, take EF, EC as axes, then, for a point Q,

$$z = a\sqrt{3} - \frac{x+y}{\sqrt{3}}\,,$$

$$r^2 = \tfrac{2}{3}(x^2+y^2-xy),$$

and the rest of the process is the same as in the first case.

(5) *A quadrant of a circle is just immersed vertically, with one edge in the surface, in a liquid, the density of which varies as the depth.*

Taking Ox as the edge, in the surface, $\rho = \mu y$ and $p = \frac{1}{2}\mu g y^2$; the centre of pressure is therefore given by the equations

$$\bar{x} = \frac{\int_0^a \int_0^{\sqrt{a^2 - x^2}} xy^2 dy\, dx}{\iint y^2 dy\, dx}, \text{ and } \bar{y} = \frac{\iint y^3 dy\, dx}{\iint y^2 dy\, dx};$$

or, in polar co-ordinates,

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} \int_0^a r^4 \sin^2\theta \cos\theta\, dr\, d\theta}{\iint r^3 \sin^2\theta\, dr\, d\theta}, \text{ and } \bar{y} = \frac{\iint r^4 \sin^3\theta\, dr\, d\theta}{\iint r^3 \sin^2\theta\, dr\, d\theta};$$

and it will be found that

$$\bar{x} = \frac{16}{15}\frac{a}{\pi} \text{ and } \bar{y} = \frac{32}{15}\frac{a}{\pi}.$$

(6) *A semicircular area completely immersed in water with its plane vertical and one end* A *of its bounding diameter in the surface.*

Let α be the inclination of the diameter to the surface, and x, y the co-ordinates of the centre of pressure referred to the diameter and the tangent at A.

Then $\quad x \iint r^2 \sin(\theta + \alpha)\, dr d\theta = \iint r^3 \cos\theta \sin(\theta + \alpha)\, dr d\theta,$

and $\quad y \iint r^2 \sin(\theta + \alpha)\, dr d\theta = \iint r^3 \sin\theta \sin(\theta + \alpha)\, dr d\theta,$

r being taken from o to $2a\cos\theta$, and θ from o to $\frac{\pi}{2}$.

38. If a given plane area turn in its own plane about a fixed point, the centre of pressure changes its position and describes a curve on the area.

Take the fixed point O as origin, and the horizontal line in the plane as axis of x'.

Let OX, OY be axes fixed in the area; then, if h be the depth of O below the surface, β the inclination of the area to the vertical, and $x'Ox = \theta$,

$$p = g\rho\,(h + y'\cos\beta) = g\rho\,(h + x\sin\theta\cos\beta + y\cos\theta\cos\beta);$$

$$\therefore\ \bar{x} = \frac{\iint px\,dx\,dy}{\iint p\,dx\,dy} = \frac{a + b\sin\theta + c\cos\theta}{\alpha + \beta\sin\theta + \gamma\cos\theta}$$

and $$\bar{y} = \frac{a' + c\sin\theta + c'\cos\theta}{\alpha + \beta\sin\theta + \gamma\cos\theta},$$

a, b, α, &c., being known constants, and the elimination of θ gives a conic section as the locus of the centre of pressure.

We can also obtain this result by aid of the theorem of Art. 36.

Taking the principal axes through G the centre of gravity as co-ordinate axes, and taking α, β as the co-ordinates of the point O, the centre of pressure is the pole of the line,

$$x\cos\theta + y\sin\theta = h + \alpha\cos\theta + \beta\sin\theta,$$

with regard to the momental ellipse, and is given by the equations,

$$\frac{a^2\cos\theta}{\bar{x}} = \frac{b^2\sin\theta}{\bar{y}} = h + \alpha\cos\theta + \beta\sin\theta.$$

If O and G coincide, the locus is $\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} = \dfrac{1}{h^2}$.

39. *A vessel having a plane base and plane vertical sides, contains two liquids which do not mix; to find the resultant pressure on one of the sides, and the centre of pressure.*

Let ρ be the density and h the depth of the upper liquid, ρ', h', corresponding quantities for the lower liquid; the common surface must be a horizontal plane, the pressure at any point of which is $g\rho h$, and the pressure at a depth z below the common surface is $g\rho h + g\rho' z$.

Taking b for the breadth of one of the vertical sides, the pressure of the upper liquid upon it $= \frac{1}{2}g\rho b h^2$, and the pressure of the lower liquid

$$= \int_0^{h'} g\,(\rho h + \rho' z)\,b\,dz = gbh'\,(\rho h + \tfrac{1}{2}\rho' h').$$

The resultant pressure is the sum of these two and is equal to

$$gb\left(\tfrac{1}{2}\rho h^2 + \rho h h' + \tfrac{1}{2}\rho' h'^2\right).$$

The moment of the fluid pressure on this side about its line of intersection with the surface

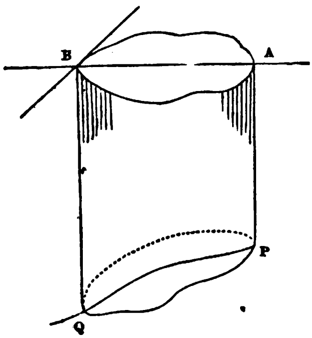
$$= \int_0^h g\rho b z^2 dz + \int_0^{h'} g(\rho h + \rho' z)\, b\,(h+z)\, dz:$$

performing the integrations, and dividing by the expression for the resultant pressure investigated above, we obtain the depth of the centre of pressure.

Resultant Pressures on Curved Surfaces.

40. *To find the resultant vertical pressure on any surface of a homogeneous liquid at rest under the action of gravity.*

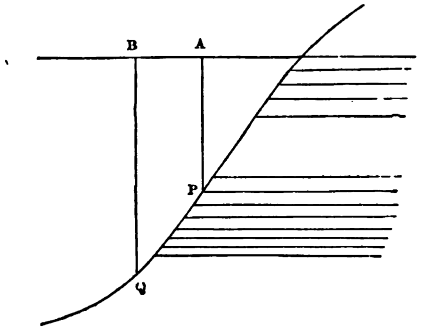
PQ being a surface exposed to the action of a heavy liquid, let *AB* be the projection of *PQ* on the surface of the liquid.



The mass *AQ* is supported by the horizontal pressure of the liquid and by the reaction of *PQ*; this reaction resolved vertically must be equal to the weight of *AQ*, and conversely, the pressure on *PQ* is equal to the weight of *AQ*, and acts through its centre of gravity.

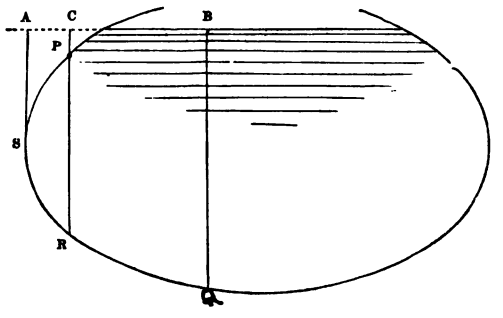
If *PQ* be pressed upwards by the liquid as in the figure, produce the surface, project *PQ* on it as before, suppose the

space AQ to be filled with liquid of the same kind, and remove the liquid from the inside.



Then the pressures at all points of PQ are the same as before, but in the contrary direction, and since the vertical pressure in this hypothetical case is equal to the weight of AQ, it follows that in the actual case, the resultant vertical pressure upwards is equal to the weight of AQ.

If the surface be pressed partially upwards and partially downwards, draw through P, the highest point of the portion of surface considered, a vertical plane PR, and let ACB be the projection of PSQ on the surface of the liquid.



Then the resultant vertical pressure on PSR,

= the weight of the liquid in PSR,

and on RQ = ... CQ,

and the whole vertical pressure = the weight of the liquid in CQ + the weight of the liquid in PR.

This might also have been deduced from the two previous articles, for PR can be divided by the line of contact of vertical tangent planes into two portions PS, SR, on which the pressures are respectively upwards and downwards; and since

$$\text{pressure on } PS = \text{weight of liquid } APS,$$
$$\text{and} \ldots\ldots\ldots\ldots\ldots \; SR = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; ASR,$$

the difference of these, i.e. the vertical pressure on PR = weight of fluid PR.

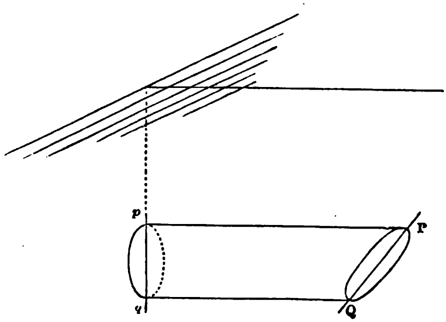
In a similar manner other cases may be discussed.

It will be observed that this investigation applies also to the case of a heterogeneous liquid (in which the density must be a function of the depth, since surfaces of equal pressure are surfaces of equal density), provided we consider that the hypothetical extension of the liquid follows the same law of density.

41. *To find the resultant horizontal pressure, in a given direction, on a surface* PQ.

Project PQ on a vertical plane perpendicular to the given direction, and let pq be the projection.

Then the mass Pq is kept at rest by the pressure on pq, the resultant horizontal pressure on PQ, and forces in vertical planes parallel to the plane pq.



Hence the horizontal pressure on PQ is equal to that on pq, and acts in the same straight line, i.e. through the centre of pressure of pq.

Hence, in general, to determine the resultant fluid pressure on any surface, find the vertical pressure, and the resultant horizontal pressures in two directions at right angles to each other. These three forces may in some cases be compounded into a single force, the condition for which may be determined by the usual methods of Statics.

Ex. *A hemisphere is filled with homogeneous liquid: required to find the resultant action on one of the four portions into which it is divided by two vertical planes through its centre at right angles to each other.*

Taking the centre O as origin, the bounding horizontal radii as axes of x and y, and the vertical radius as the axis of z, the pressure parallel to x is equal to the pressure on the quadrant yOz, which is the projection, on a plane perpendicular to Ox, of the curved surface.

Therefore, the pressure parallel to Ox

$$=g\rho\,\frac{\pi a^2}{4}\cdot\frac{4a}{3\pi}=\frac{1}{3}g\rho a^3,$$

and the co-ordinates of its point of action are

$$\left(o,\ \frac{3}{8}a,\ \frac{3}{16}\pi a\right),\ \text{Art. 37, Ex. 1;}$$

similarly, the pressure parallel to $Oy=\frac{1}{3}g\rho a^3$, and acts through the point,

$$\left(\frac{3}{8}a,\quad o,\quad \frac{3}{16}\pi a\right).$$

The resultant vertical pressure $=$ the weight of the liquid $=\frac{1}{6}g\rho\pi a^3$, and acts in the direction of the line $x=\frac{3}{8}a=y$.

The directions of the three forces all pass through the point

$$\left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{16}\pi a\right),$$

and they are therefore equivalent to a single force

$$\frac{1}{6}g\rho a^3\sqrt{(\pi^2+8)} \text{ in the line}$$

$$x-\frac{3}{8}a=y-\frac{3}{8}a=\frac{2}{\pi}\left(z-\frac{3}{16}\pi a\right),$$

$$\text{or}\quad x=y=\frac{2}{\pi}z,$$

a straight line through the centre, as must obviously be the case, since all the fluid pressures are normal to the surface. The point in which it meets the surface of the hemisphere may be called 'the centre of pressure.'

42. *To find the resultant pressure on the surface of a solid either wholly or partially immersed in a heavy liquid.*

Suppose the solid removed, and the space it occupied filled with liquid of the same kind; the resultant pressure upon it will be the same as upon the original solid. But the liquid mass is at rest under the action of its own weight, and the pressure of the liquid surrounding it: the resultant pressure is therefore equal to the weight of the liquid displaced, and acts in a vertical line through its centre of gravity.

The same reasoning evidently shews that the resultant pressure of an elastic fluid on any solid is equal to the weight of the elastic fluid displaced by the solid.

This result may also be obtained by means of Arts. 40 and 41, as follows: Draw parallel horizontal lines touching the surface, and forming a cylinder which encloses it; the curve of contact divides the surface into two parts, on which the resultant horizontal pressures, parallel to the axis of the cylinder, are equal and opposite; the horizontal pressures on the solid therefore balance each other and the resultant is

wholly vertical. To determine the amount of the resultant vertical pressure, draw parallel vertical lines touching the surface, and dividing it into two portions on one of which the resultant vertical pressure acts upwards, and on the other downwards; the difference of the two is evidently the weight of the fluid displaced by the solid.

43. If a solid of given volume (V) be completely immersed in a heavy liquid, and if the surface of the solid consist partly of a curved surface, and partly of known plane areas; the resulting pressure on the curved surface can be determined.

For the plane areas being known in size and position, we can calculate the resultant horizontal and the resultant vertical pressure, X and Y, upon those areas; and, since the resulting pressure on the whole surface is vertical and equal to $g\rho V$ upwards, it follows that the resultant horizontal and vertical pressures on the curved surface are respectively equal to X and $g\rho V - Y$.

Ex. *A solid is formed by turning a circular area round a tangent line through an angle θ, and this solid is held under water with its lower plane face horizontal and a given depth* h.

In this case,

$$V = \pi a^3\theta,\ X = g\rho\pi a^2 (h - a\sin\theta)\sin\theta,$$

and $$Y = g\rho\pi a^2 (h - h\cos\theta + a\sin\theta\cos\theta).$$

44. *To find the resultant pressure on any surface of a fluid at rest under the action of any given forces.*

Let p be the pressure, determined as in Chapter II., at any point (x, y, z) of a surface, $u = 0$, exposed to the action of the fluid. Then if

$$\frac{1}{P^2} = \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2,$$

$$P\frac{du}{dx},\ P\frac{du}{dy},\ P\frac{du}{dz},$$

are the direction-cosines of the normal at the point (x, y, z).

Let δS be an element of the surface about the same point. The pressures on this element, parallel to the axes, are

$$pP\frac{du}{dx}\delta S, \quad pP\frac{du}{dy}\delta S, \quad pP\frac{du}{dz}\delta S;$$

$\therefore$ if X, Y, Z, and L, M, N, be the resultant pressures parallel to the axes, and the resultant couples, respectively,

$$X=\iint pP\frac{du}{dx}dS, \quad Y=\iint pP\frac{du}{dy}dS, \quad Z=\iint pP\frac{du}{dz}dS,$$

$$L=\iint pPdS\left(y\frac{du}{dz}-z\frac{du}{dy}\right),$$

$$M=\iint pPdS\left(z\frac{du}{dx}-x\frac{du}{dz}\right),$$

$$N=\iint pPdS\left(x\frac{du}{dy}-y\frac{du}{dx}\right);$$

the integrations being made to include the whole of the surface under consideration.

These resultants are equivalent to a single force if

$$XL+YM+ZN=0.$$

45. The surface may be divided into elements in three different ways by planes parallel to the co-ordinate planes.

Thus, $\delta x\delta y =$ projection of δS on $xy = P\dfrac{du}{dz}\delta S$;

and $\therefore$ $Z=\iint pdxdy$; and similarly, $X=\iint pdydz$, and $Y=\iint pdzdx$,

$$L=\iint p\,(ydxdy-zdzdx),$$

$$=\iint p\,(ydy-zdz)\,dx,$$

$$M=\iint p\,(zdz-xdx)\,dy,$$

$$N=\iint p\,(xdx-ydy)\,dz.$$

46. If the fluid be at rest under the action of gravity only, and the axis of z be vertical, p is a function of z, $\phi(z)$ suppose, and therefore,

$$X=\iint \phi(z)\,dydz,$$

which is evidently the expression for the pressure, parallel to x, upon the projection of the given surface on the plane yz; and similarly Y is equal to the pressure upon the projection on xz.

Again, if the fluid be incompressible and acted upon by gravity only, $p\delta x\delta y$ is equal to the weight of the portion of fluid contained between δS and its projection on the surface of the fluid;

$\therefore$ Z, or $\iint p dx dy$, is the weight of the superincumbent fluid.

These results accord with those previously obtained, Arts. 40 and 41.

47. If a solid body be wholly or partially immersed in any fluid which is at rest under the action of given forces, the resultant fluid pressure on the body will be equal to the resultant of the forces which would act on the displaced fluid.

For we can imagine the solid removed and the gap filled up with the fluid, which will be in equilibrium under the action of the forces and the pressure of the surrounding fluid; and the resultant pressure must be equal and opposite to the resultant of the forces.

In filling up the gap with fluid, the law of density must be maintained, that is, the surfaces of equal density must be continuous with those of the surrounding fluid.

EXAMPLES.

1. A heavy thick rope, the density of which is double the density of water, is suspended by one end, outside the water, so as to be partly immersed; find the tension of the rope at the middle of the immersed portion.

2. Water is poured into a hollow sphere, determine the depth of the water when the resultant pressure is half the total normal pressure.

3. A conical wine-glass is filled with water and placed in an inverted position on a table; if the whole pressure of

the water on the glass be double its resultant pressure, find the vertical angle of the cone.

4. A vessel in the form of a regular pyramid, whose base is a plane polygon of n sides, is placed with its axis vertical and vertex downwards and is filled with fluid. Each side of the vessel is moveable about a hinge at the vertex, and is kept in its place by a string fastened to the middle point of its base and to the centre of the polygon: shew that the tension of each string is to the whole weight of the fluid as 1 to $n \sin 2\alpha$, where α is the inclination of each side to the horizon.

5. Find the centre of pressure of a square lamina having one angular point in the surface of a liquid; and supposing it to be moved about the angular point in its own plane, which is fixed, and to be always totally immersed, find the locus on its own plane of its centre of pressure.

6. Find the centre of pressure of an elliptic lamina just immersed in water; and supposing it turned round in the same vertical plane, so as to be always just immersed, find the locus with respect to its axes of the centre of pressure.

7. A cubical box, filled with water, has a close-fitting heavy lid fixed by smooth hinges to one edge; compare the tangents of the angles through which the box must be tilted about the several edges of its base, in order that the water may just begin to escape.

8. A system of coaxal circles is immersed in water with the line of centres at a given depth; prove that the centres of pressure of those circular areas, which are completely immersed, lie on a parabola.

9. Find the centre of pressure of a semi-ellipse (axes $2a$ and a) which is bounded by a diameter inclined at the angle $\frac{\pi}{6}$ to its major axis, its plane being vertical, and the diameter in the surface.

10. A semi-ellipse bounded by its axis minor, is just immersed in a liquid the density of which varies as the depth;

if the axis minor be in the surface, find the eccentricity in order that the focus may be the centre of pressure.

11. A square lamina $ABCD$, which is immersed in water, has the side AB in the surface; draw a line BE to a point E in CD such that the pressures on the two portions may be equal. Prove that, if this be the case, the distance between the centres of pressure : the side of the square :: $\sqrt{505}$: 48.

12. From a semicircle, whose diameter is in the surface of a liquid, a circle is cut out, whose diameter is the vertical radius of the semicircle; find the centre of pressure of the remainder.

13. A semicircular lamina is completely immersed in water with its plane vertical, so that the extremity A of its bounding diameter is in the surface, and the diameter makes with the surface an angle α.

Prove that if E be the centre of pressure and θ the angle between AE and the diameter,

$$\tan\theta = \frac{3\pi + 16\tan\alpha}{16 + 15\pi\tan\alpha}.$$

14. If the depths of the angular points of a triangle below the surface of a liquid be a, b, c, prove that the depth of the centre of pressure below the centre of gravity is

$$\frac{(b-c)^2 + (c-a)^2 + (a-b)^2}{12(a+b+c)}.$$

15. A plane area immersed in a fluid moves parallel to itself and with its centre of gravity always in the same vertical straight line. Shew (1) that the locus of the centres of pressure is a hyperbola, one asymptote of which is the given vertical, and (2) that if a, $a+h$, $a+h'$, $a+h''$, be the depths of the C.G. any positions, y, $y+k$, $y+k'$, $y+k''$, those of the centre of pressure in the same positions, then

$$\begin{vmatrix} k & h & h\,(k-h) \\ k' & h' & h'\,(k'-h') \\ k'' & h'' & h''(k''-h'') \end{vmatrix} = 0.$$

16. Find the centre of pressure of a segment of a parabola bounded by the curve and the latus-rectum, the tangent at one end of the bounding ordinate being in the surface. If the liquid rise, the parabola remaining stationary, shew that the centre of pressure describes a straight line.

17. A cone is totally immersed in water, the depth of the centre of its base being given. Prove that, P, P', P'', being the resultant pressures on its convex surface, when the sines of the inclination of its axis to the horizon are s, s', s'', respectively,

$$P^2(s'-s'')+P'^2(s''-s)+P''^2(s-s')=0.$$

18. Find the centre of pressure of the area between the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$, and the axes, taking the axes rectangular and one of them in the surface.

19. A quantity of liquid acted upon by a central force varying as the distance is contained between two parallel planes; if A, B, be the areas of the planes in contact with the fluid, shew that the pressures upon them are in the ratio $A^2 : B^2$.

20. A hollow sphere is full of liquid, the density of which varies as (the depth)^n; shew that the whole pressure on the surface of the sphere : the resultant pressure :: $n+3 : n+1$.

21. One asymptote of a hyperbola lies in the surface of a fluid; find the depth of the centre of pressure of the area included between the immersed asymptote, the curve, and two given horizontal lines in the plane of the hyperbola.

22. A cone is immersed in water with the centre of its base at a distance of $\frac{5}{6}$ of its altitude below the surface. A paraboloid of the same base and altitude is also immersed with the centre of its base at the same distance below the surface as that of the cone, and with its axis inclined at the same angle to the vertical. Find what this angle must be in order that the resultant pressures on the convex surfaces of the two solids may be equal.

23. A closed cylinder, very nearly filled with liquid, rotates uniformly about a generating line, which is vertical; find the resultant pressure on its curved surface.

Determine also the point of action of the pressure on its upper end.

24. Shew that the depth of the centre of pressure of the area included between the arc and the asymptote of the curve $(r-a)\cos\theta = b$ is $\frac{a}{4}\cdot\frac{3\pi a+16b}{3\pi b+4a}$, the asymptote being in the surface and the plane of the curve vertical.

25. A cone is filled with liquid, and fitted with a heavy lid, moveable about a hinge; it is then made to revolve uniformly about the generating line through the hinge, which is vertical; find the greatest angular velocity consistent with no escape of the liquid.

26. A portion of a spherical shell is cut off by a plane, and the remaining portion is placed on a horizontal plane so that the circular section is in contact with the plane and is then filled with water through a small hole at the highest point. Find the largest piece which can be cut off so that, however light the shell may be, the water may not escape.

In this case, prove that the whole pressure on the shell is to the weight of the liquid in the ratio 2 : 1.

27. If a plane area immersed in a liquid revolve about any axis in its own plane, prove that the centre of pressure describes a straight line in the plane.

28. A cube whose edge is $2a$, and whose faces are horizontal and vertical, is surrounded by a mass of heavy liquid, the volume of which is $8a^3\{\pi\sqrt{6}-1\}$; the liquid is acted on by a force tending to the centre of the cube, and varying as the distance, the force at the distance a being g: find the form of the free surface and the pressure at any point: also if one of the vertical faces of the cube be moveable about a horizontal line in its own plane, shew that the face will be at rest, if this line be at a distance $\frac{4}{5}a$ from the lowest edge of that face.

29. A solid paraboloid, cut off by a plane through the focus perpendicular to its axis, is completely immersed, its vertex being at a given depth, and its axis inclined at a given angle to the vertical. Find the direction and magnitude of the resultant pressure on its curved surface.

30. A solid is formed by turning a parabolic area, bounded by the latus-rectum, about the latus-rectum, through an angle θ; and this solid is held under water, just immersed, with its lower plane face horizontal. Prove that, if ϕ be the inclination to the horizon of the resultant pressure on the curved surface of the solid,

$$3 \sin^2 \theta \tan \phi = 5 \sin \theta - 3 \sin \theta \cos \theta - 2\theta.$$

31. In the midst of a mass of fluid attracting according to the law of nature, and rotating in relative equilibrium about an axis, a small particle is introduced, and started with the velocity of the fluid whose place it occupies. Will it approach or recede from the axis?

32. In an infinite mass of fluid of density ρ, every part of which attracts every other part according to the law of nature, are placed two shells, whose internal and external radii are a, b and a', b' respectively, and densities σ, σ'. The shells also attract each other and the fluid as in nature. Find the resultant force on each shell, and shew that in certain cases this force is a repulsive one.

33. A given area is immersed vertically in a heavy liquid and a cone is constructed on it as base, the cone being wholly immersed: find the locus of the vertex when the resultant pressure on the curved surface is constant, and shew that this pressure is unaltered by turning the cone round the horizontal line drawn through the centre of gravity of the base perpendicular to the plane of the base.

34. A conical vessel, axis vertical and vertex downwards, is divided into two parts by a plane through its axis, and the parts are prevented from separating by a string which is a diameter of the rim of the vessel, and is perpendicular to the dividing plane, and by a hinge at the vertex.

The vessel being filled with water, compare the tension of the string with the weight of the water.

35. A hollow cone open at the top is filled with water; find the resultant pressure on the portion of its surface cut off, on one side, by two planes through its axis inclined at a given angle to each other; also determine the line of action of the resultant pressure, and shew that, if the vertical angle be a right angle, it will pass through the centre of the top of the cone.

36. A bowl in the form of a hemisphere is filled with water; find the direction and magnitude of the resultant pressure on the upper portion of the bowl cut off by a plane through its centre inclined at a given angle to the horizon.

37. An open conical shell, the weight of which may be neglected, is filled with water, and is then suspended from a point in the rim, and allowed gradually to take its position of equilibrium; prove that, if the vertical angle be $\cos^{-1}\frac{2}{3}$, the surface of the water will divide the generating line through the point of suspension in the ratio 2 : 1.

38. A regular polygon wholly immersed in a liquid is moveable about its centre of gravity; prove that the locus of the centre of pressure is a sphere.

39. A hemispherical bowl is filled with water, and two vertical planes are drawn through its central radius, cutting off a semi-lune of the surface; if 2α be the angle between the planes, prove that the angle which the resultant pressure on the surface makes with the vertical

$$=\tan^{-1}\left(\frac{\sin\alpha}{\alpha}\right).$$

40. A vessel in the form of a surface of revolution has the following property; if it be placed with its axis vertical, and any quantity of water be poured into it, the ratio of the total normal pressure to the resultant vertical pressure varies

as the depth of the water poured in. Shew that the equation to the generating curve is

$$cs = xy.$$

41. Find the equation of a curve symmetrical about a vertical axis, such that, when it is immersed with its highest point at half the depth of its lowest, the centre of pressure may bisect the axis.

CHAPTER IV.

THE EQUILIBRIUM OF FLOATING BODIES.

48. *To find the conditions of equilibrium of a floating body.*

We shall suppose that the fluid is at rest under the action of gravity only, and that the body, under the action of the same force, is floating freely in the fluid. The only forces then which act on the body are its weight, and the pressure of the surrounding fluid, and in order that equilibrium may exist, the resultant fluid pressure must be equal to the weight of the body, and must act in a vertical direction.

Now we have shewn, that the resultant pressure of a heavy fluid on the surface of a solid, either wholly or partially immersed, is equal to the weight of the fluid displaced, and acts in a vertical line through its centre of gravity.

Hence it follows that the weight of the body must be equal to the weight of the fluid displaced, and that the centres of gravity of the body, and of the fluid displaced, must lie in the same vertical line.

These conditions are necessary and sufficient conditions of equilibrium, whatever be the nature of the fluid in which the body is floating. If it be heterogeneous, the displaced fluid must be looked upon as following the same law of density as the surrounding fluid; in other words, it must consist of strata of the same kind as, and continuous with, the horizontal strata of uniform density, in which the particles of the surrounding fluid are necessarily arranged.

If for instance a solid body float in water, partially immersed, its weight will be equal to the weight of the water displaced, together with the weight of the air displaced; and if the air be removed, or its pressure diminished by a diminution of its density or temperature, the solid will sink in the water through a space depending upon its own weight, and upon the densities of air and water. This may be further explained by observing that the pressure of the air on the water is greater than at any point above it, and that this surface pressure of the air is transmitted by the water to the immersed portion of the floating body, and consequently the upward pressure of the air upon it is greater than the downward pressure.

49. We now proceed to illustrate the application of the above conditions, by the discussion of some particular cases.

Ex. 1. *A portion of a solid paraboloid, of given height, floats with its axis vertical and vertex downwards in a homogeneous liquid, required to find its position of equilibrium.*

Taking $4a$ as the latus rectum of the generating parabola, h its height, and x the depth of its vertex, the volumes of the whole solid and of the portion immersed are respectively $2\pi ah^2$ and $2\pi ax^2$; and if ρ, σ, be the densities of the solid and liquid, one condition of equilibrium is

$$\rho \, . \, 2\pi ah^2 = \sigma \, . \, 2\pi ax^2;$$

$$\therefore \; x = \sqrt{\frac{\rho}{\sigma}}\, h,$$

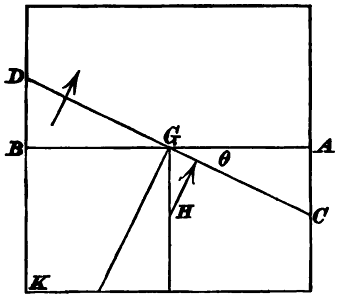
which determines the portion immersed, the other condition being obviously satisfied.

Ex. 2. *It is required to find the positions of equilibrium of a square lamina floating with its plane vertical, in a liquid of double its own density.*

The conditions of equilibrium are clearly satisfied if the lamina float half immersed either with a diagonal vertical, or with two sides vertical.

To examine whether there is any other position of equilibrium, let the lamina be held with the line DGC in the surface, in which case the first condition is satisfied.

But, if the angle $CGA = \theta$, and if $2a$ be the side of the square, the moment about G of the fluid pressure, which is the same as the difference between the moments of the rectangle AK, and of twice the triangle GBD,



$$\propto 2a^2 \cdot \frac{a}{2} \sin\theta - a^2 \tan\theta \cdot \frac{a\sec\theta + a\cos\theta}{3},$$

$$\propto \sin\theta\,(1 - \tan^2\theta);$$

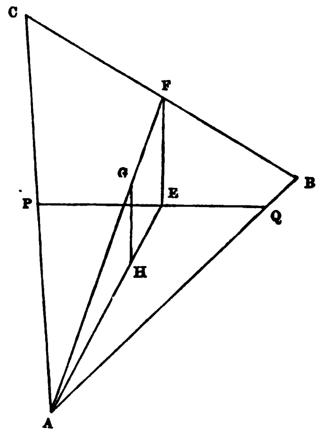
and this vanishes only when $\theta = 0$ or $\frac{\pi}{4}$.

Hence there is no other position of equilibrium.

Ex. 3. *A triangular prism floats with its edges horizontal, to find its positions of equilibrium.*

Let the figure be a section of the prism by a vertical plane through its centre of gravity.

PQ is the line of floatation and H the centre of gravity of the liquid displaced. When there is equilibrium the area APQ is to ABC in the ratio of the density of the prism to the density of the liquid, and therefore for all possible positions of PQ the area APQ is constant; hence PQ always touches, at its middle point, an hyperbola of which AB, AC, are the asymptotes.



Also HG must be perpendicular to PQ, and therefore since

$$AH : HE = AG : GF,$$

FE must be perpendicular to PQ, that is, FE is the normal at E to the hyperbola. The problem is therefore reduced to that of drawing normals from F to the curve.

Let $xy = c^2$ be the equation of the curve referred to AB, AC as axes, and let

$$\angle BAC = \theta,\ AB = 2a,\ AC = 2b. \ldots\ldots\ldots(\alpha).$$

Let x, y, be the co-ordinates of E; the co-ordinates of F are a, b, and the equation of the normal at E is

$$\eta - y = \frac{y\cos\theta - x}{x\cos\theta - y}(\xi - x).$$

And if this pass through F, the co-ordinates of which are a, b,

$$(b - y)(x\cos\theta - y) = (a - x)(y\cos\theta - x),$$

$$\text{or } x^2 - (a + b\cos\theta)\,x = y^2 - (a\cos\theta + b)\,y \ldots\ldots(\beta).$$

The equations (α) and (β) determine all the points of the hyperbola, the tangents at which can be lines of floatation.

Also (β) is the equation to an equilateral hyperbola, referred to conjugate diameters parallel to AB, AC; the points of intersection of the two hyperbolas are therefore the positions of E.

To find x, we have

$$x^4 - (a + b\cos\theta) \,.\, x^3 + (a\cos\theta + b)\, c^2 x - c^4 = 0,$$

an equation which has only one negative root, and one or three positive roots, and there may be therefore three positions of equilibrium or only one.

If the densities of the liquid and the prism be ρ and σ, we have, since the area PAQ

$$= \tfrac{1}{2}\, AP \,.\, AQ \sin\theta = 2xy \sin\theta = 2c^2 \sin\theta,$$

$$2\rho c^2 \sin\theta = 2\sigma ab \sin\theta,$$

$$\text{or}\quad \rho c^2 = \sigma ab,$$

from which c is determined.

Suppose the prism to be isosceles, then putting $a = b$, the equation for x becomes

$$x^4 - c^4 - a(1 + \cos\theta)(x^3 - c^2 x) = 0;$$

from which we obtain $x = c$, which gives $y = c$, and makes BC horizontal, an obvious position of equilibrium, and also

$$x = \frac{a}{2}(1 + \cos\theta) \pm \left\{\frac{a^2}{4}(1 + \cos\theta)^2 - c^2\right\}^{\frac{1}{2}}$$

$$= a\cos^2\frac{\theta}{2} \pm \left(a^2\cos^4\frac{\theta}{2} - c^2\right)^{\frac{1}{2}};$$

the isosceles prism will therefore have only one position of equilibrium, unless

$$a\cos^2\frac{\theta}{2} > c;$$

and, since $\rho c^2 = \sigma a^2$, this is equivalent to

$$\cos^2\frac{\theta}{2} > \sqrt{\frac{\sigma}{\rho}}.$$

Ex. 4. *Determine the position of equilibrium of a balloon of given size and weight, neglecting the variations of temperature at different heights in the atmosphere.*

If the temperature be constant, the pressure of the air at a height $z = \Pi\epsilon^{-\frac{gz}{k}}$, and its density $= \frac{\Pi}{k}\epsilon^{-\frac{gz}{k}}$, Π being the atmospheric pressure at the level from which the height is measured.

The air displaced consists of a series of strata of variable density, and if z be the height of the lowest point of the balloon, x the distance from that point of any horizontal section (X) of the balloon, and h its height, the weight of a stratum of the air displaced is

$$\frac{\Pi g}{k}\epsilon^{-\frac{g(z+x)}{k}} X\delta x,$$

and the whole weight of air displaced

$$= \int_0^h \frac{\Pi g}{k}\epsilon^{-\frac{g(z+x)}{k}} X dx = \frac{\Pi g}{k}\epsilon^{-\frac{gz}{k}}\int_0^h \epsilon^{-\frac{gx}{k}} X dx.$$

The form of the balloon being given, X is a known function of x, and if W be the weight of the balloon and of the gas it contains, the height z will be determined by equating W to the expression we have obtained for the weight of the air displaced.

50. *A homogeneous solid floats, wholly immersed, in a liquid of which the density varies as the depth; to find the depth of its centre of gravity.*

Let a, c, be the depths of the highest and lowest points of the solid, Z the area of a horizontal section of the solid at a depth z, and μz the density;

$$\text{the weight of the liquid displaced} = \int_a^c g\mu z Z dz.$$

Let $\bar{z}$ be the depth of the centre of gravity of the solid, and V its volume, then

$$V\bar{z} = \int_a^c Z z dz;$$

therefore the weight of displaced liquid $= g\mu\bar{z}V$, and if ρ be the density of the solid, its weight $= g\rho V$; hence $\rho = \mu\bar{z}$, or the solid floats in such a position that the density of the liquid at the depth of the centre of gravity of the solid is equal to the density of the solid.

51. If a solid float under constraint, the conditions of equilibrium depend on the nature of the constraining circumstances, but in any case the resultant of the constraining forces must act in a vertical direction, since the other forces, the weight of the body, and the fluid pressure, are vertical.

If for instance one point of a solid be fixed, the condition of equilibrium is that the weight of the body and the weight of the fluid displaced should have equal moments about the fixed point; this condition being satisfied, the solid will be at rest, and the stress on the fixed point will be the difference of the two weights.

As an additional illustration, consider the case of a solid floating in water and supported by a string fastened to a point above the surface; in the position of equilibrium the string will be vertical, and the tension of the string, together with the resultant fluid pressure, which is equal to the weight of the displaced fluid, will counterbalance the weight of the body; the tension is therefore equal to the difference of the weights, and the weights are inversely in the ratio of the distances of their lines of action from the line of the string, these three lines being in the same vertical plane.

52. For subsequent investigations, the following geometrical propositions will be found important.

If a solid be cut by a plane, and this plane be made to turn through a very small angle about a straight line in itself, the volume cut off will remain the same, provided the straight line pass through the centre of gravity of the area of the plane section.

To prove this, consider a right cylinder of any kind cut by a plane making with its base an angle θ.

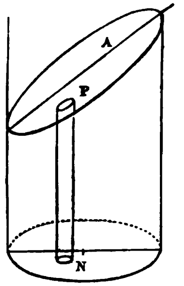
Let $\bar{z}$ be the distance from the base of the centre of gravity of the section A, δA an element of the area of the section and V the volume between the planes. Then

$$\bar{z} = \frac{\Sigma(\delta A \,.\, PN)}{A};$$

$$\therefore \; A \cos\theta \bar{z} = \Sigma(\delta A \cos\theta \,.\, PN) = V,$$

$$\text{or } \; V = \bar{z}\,(\text{area of base}).$$

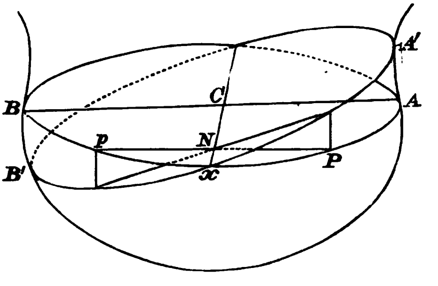
Now the centre of gravity of the area A is also the centre of gravity of all sections made by planes passing through it, as may be seen by projecting the sections on the base of the cylinder; it follows therefore, that, $\bar{z}$ being the same for all such sections, the volumes cut off are the same.



In the case of any solid, if the cutting plane be turned through a very small angle about the centre of gravity of its section, the surface near the curves of section may be considered, without sensible error, cylindrical, and the above proposition is therefore established*.

* The following form of proof may also be given.

Let ACB, the cutting plane, be turned through a small angle (θ) about a line Cx, and let dA be an element of the area.



Then the algebraical value of the additional volume cut off is equal to $\int \theta y dA$, and, if this vanishes, $\int y dA = 0$, which is the condition that the centre of gravity of A should lie in the axis of x; and, taking C as the centre of gravity, any plane through C will satisfy the same condition.

In other words, the difference between the volume lost and the volume gained by the change in the position of the cutting plane will be indefinitely small compared with either.

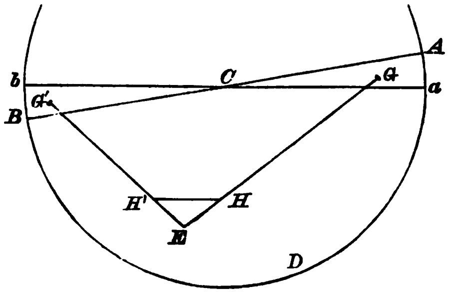
53. *Definitions.* If a body float in a homogeneous liquid, the plane in which the body is intersected by the surface of the liquid is the *plane of floatation.*

The point H, the centre of gravity of the liquid displaced, is the *centre of buoyancy.*

If the body move so that the volume of liquid displaced remains unchanged, the envelope of the planes of floatation is the *surface of floatation,* and the locus of H is the *surface of buoyancy.*

54. *If a plane move so as to cut from a solid a constant volume, and if H be the centre of gravity of the volume cut off, the tangent plane at H to the surface which is the locus of H is parallel to the cutting plane.*

In other words, the tangent planes at any point of the



surface of floatation, and at the corresponding point of the surface of buoyancy are parallel to one another.

We may observe that the algebraical moment about the axis of y of the volume cut off is $\int \theta xy dA$, which vanishes if $\int xy dA = 0$, that is, if the axes Cx, Cy be the principal axes of the area.

Turn the plane ACB, the cutting plane, through a small angle into the position aCb, the volumes of the wedges ACa, BCb being equal.

Let G and G' be the centres of gravity of these wedges.

In GH produced take a point E such that

$$EH : HG :: \text{Volume } ACa : \text{Volume } aDB.$$

Join EG' and take H' such that

$$EH' : H'G' :: \text{Vol. } BCb : \text{Vol. } aDB;$$

then H' is the centre of gravity of aDb;

$$\therefore EH : HG :: EH' : H'G',$$

and HH' is therefore parallel to GG'.

Hence it follows that ultimately when the angle ACa is indefinitely diminished,

$$HH' \text{ is parallel to } ACB;$$

and HH' is a tangent at H to the locus of H.

This being true for any displacement of the plane ACB about its centre of gravity, it follows that the tangent plane at H to the locus of H is parallel to the plane ACB.

55. *The positions of equilibrium of a body floating in a homogeneous liquid are determined by drawing normals from G, the centre of gravity of the body, to the surface of buoyancy.*

For if GH be a normal to the surface of buoyancy, the tangent plane at H, being parallel to the plane of floatation, is horizontal, and GH is therefore vertical.

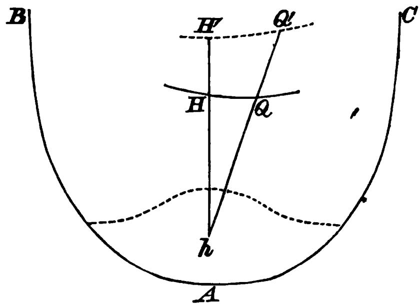
The two conditions of equilibrium are then satisfied, and a position of equilibrium is determined.

The problem comes to the same thing as determining the positions of equilibrium of a heavy body, bounded by the surface of buoyancy, on a horizontal plane.

56. It should be noticed that the shape of the curve of buoyancy is entirely determined by the form of the bounding

surface, and is unaffected by an alteration of the form of that portion of the body which always remains immersed. ?

Let HQ be an arc of the surface of buoyancy for a boundary BAC, and an immersed volume V, and imagine a volume v, the centre of gravity of which is h, to be cut off.



Taking $hH' : hH :: hQ' : HQ :: V : V-v$, the surface $H'Q'$ is the new curve of buoyancy which is obviously similar to the surface HQ.

57. *Particular cases of curves of buoyancy.*

For a triangular prism, as in Art. 49, the curve of floatation is the envelope of PQ, which is an hyperbola having AB, AC for asymptotes; and, since $AH = \frac{2}{3}AE$, the curve of buoyancy is a similar hyperbola.

If the body be a plane lamina bounded by a parabola, the curves of floatation and buoyancy are equal parabolas.

If the boundary be an elliptic arc, the curves are arcs of similar and similarly situated concentric ellipses.

If the immersed portion of a lamina (or prism), be a rectangle, the curve of floatation apparently is a single point, and the curve of buoyancy is a parabola.

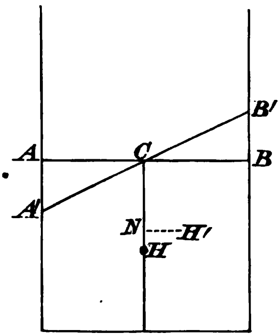
To prove this, let H, H' be positions of the centre of gravity corresponding to the positions ACB, $A'CB'$ of the line of floatation.

Then, if $AC = CB = a$, $BB' = \beta$, $CH = c$, and $S =$ the area cut off,

$$Sy = S \,.\, H'N = \tfrac{1}{2}a\beta \,.\, \frac{2a}{3} - \tfrac{1}{2}a\beta\left(-\frac{2a}{3}\right) = \tfrac{2}{3}a^2\beta,$$

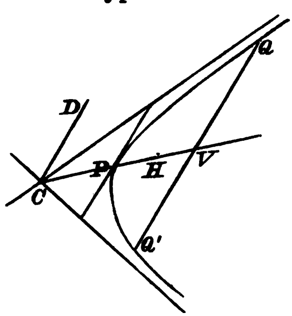
$$Sx = S \,.\, HN = \tfrac{1}{2}a\beta\left(c + \frac{\beta}{3}\right) - \tfrac{1}{2}a\beta\left(c - \frac{\beta}{3}\right) = \tfrac{1}{3}a\beta^2,$$

and $$\therefore\ Sy^2 = \tfrac{4}{3}a^3x.$$



This is a particular case of the triangular prism, and, as in that case, the curves of floatation and buoyancy are similar curves, the fact being that the curve of floatation is a parabola, with its vertex at C, flattened down to a straight line. The identity of the cases will be clearly seen by the application of a powerful microscope to the figure, capable of enlarging the evanescent parabola to a visible size.

58. If the body be a lamina bounded by an hyperbolic arc, the curves are similar hyperbolas.



Thus, if QVQ' be a line of floatation, and if $2a', 2b'$ be the diameters conjugate and parallel to QQ', inclined at an angle θ, so that $a'b' \sin\theta = ab$,

$$\text{the area } QVQ' = 2\int_{a'}^{x'} \frac{b'}{a'} \sqrt{x^2 - a'^2} \sin\theta dx$$

$$= ab\left\{\frac{x'}{a'}\sqrt{\frac{x'^2}{a'^2} - 1} + \log\left(\frac{x'}{a'} + \sqrt{\frac{x'^2}{a'^2} - 1}\right)\right\},$$

so that the ratio of x' to a', that is, of CV to CP is constant.

Moreover,

$$(\text{area})\ (CH) = 2\frac{b'}{a'} \sin\theta \int x\sqrt{x^2 - a'^2}\, dx'$$

$$= \frac{2}{3}ab\left(\frac{x'^2}{a'^2} - 1\right)^{\frac{3}{2}} a';$$

and therefore the ratio of CH to CP is constant.

These results can also be obtained by purely geometrical reasoning.

59. If the floating body be such that the boundary of the portion immersed is the surface of an ellipsoid, it is easily seen that the surfaces of floatation and buoyancy are portions of similar and similarly situated concentric ellipsoids. For if the boundary be a portion of a spherical surface, this is obviously true, and the sphere can be homogeneously strained into an ellipsoid.

60. *A solid of revolution floats in a liquid which rotates uniformly, as if solid, about a vertical axis, the axis of the solid coinciding with the axis of rotation; required to find the condition of equilibrium.*

In a mass of rotating liquid, suppose a surface of revolution described, having its axis coincident with the axis of rotation, and consider the equilibrium of the liquid within this surface. The resultant of the fluid pressures upon the liquid must be equal to its weight, and the same pressures

being exerted on the surface of any solid occupying the same space, it follows that any such solid will be in equilibrium, if its weight be equal to the weight of the fluid it displaces.

It will be seen moreover that it is quite indifferent whether the solid rotate with the fluid, or with a different angular velocity, or be at rest.

Ex. *A cylinder floats in rotating liquid; to find the depth to which it is immersed.*

If ω be the angular velocity, the equation to the generating parabola of the free surface, taking its vertex as the origin, is $\omega^2 y^2 = 2gz$, and if z be the depth of the base of the cylinder below the circle of floatation, that is, the circle in which the free surface intersects the surface of the cylinder, and c the radius of the cylinder, the volume of the displaced fluid is the difference between the volume of a height z of the cylinder, and the volume of a height $\frac{\omega^2 c^2}{2g}$ of the paraboloid.

Hence, if σ be the density of the cylinder and ρ of the fluid,

$$\sigma\pi c^2 h = \rho\left(\pi c^2 z - \frac{\pi\omega^2 c^4}{4g}\right),$$

$$\text{and } z = \frac{\sigma}{\rho} h + \frac{\omega^2 c^2}{4g}.$$

61. A more general case is that of a body floating, wholly or partially immersed, in a liquid at rest under the action of any given forces, the same forces being supposed to act on the molecules of the body.

If the body be in equilibrium, the resulting force upon it will be equal to the resulting force on the liquid displaced, and the lines of action of the two forces will be the same.

For, if the body be removed, and its place occupied by the displaced liquid, the resulting pressure of the liquid upon the body will be the same as upon the displaced liquid, and will therefore be equal and opposite to the resultant force upon the displaced liquid.

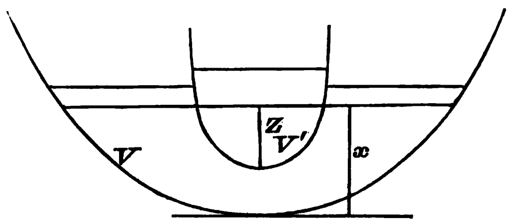
Ex. *A mass of liquid is at rest under the action of a force to a fixed point varying as the distance, and a solid in the form of a spherical sector is at rest partly immersed in it, with its vertex at the fixed point; it is required to compare the densities of the liquid and the solid.*

In the state of equilibrium, let r be the radius of the free surface of the liquid, and a the radius of the spherical sector. The volumes of the sector and of the displaced liquid are in the ratio of a^3 to r^3; and the distances of their centre of gravity from the centre of force are in the ratio of a to r;

$$\therefore \text{ if } \rho \text{ and } \sigma \text{ be the densities, } \rho a^4 = \sigma r^4.$$

62. *Potential energy stored up by the immersion of a solid in a liquid.*

If a solid body be immersed in a vessel containing liquid, work is done, and therefore potential energy is gained by the elevation of the centre of gravity of the liquid.



Let x be the depth of liquid, z the depth of immersion of the solid, X and Z the corresponding areal sections of the vessel and the solid, V the volume of liquid, and V' of the immersed portion of the solid.

Then,

$$V\bar{x} = \int_0^x X'x'dx' - \int_0^z Z'(x - z + z')dz',$$

and the increase of potential energy is the variation of the expression $g\rho V\bar{x}$, due to an increase δx in x.

Taking $g\rho = 1$, this variation

$$= Xx\delta x - (\delta x - \delta z) V' - (x - z) Z\delta z - Zz\delta z,$$

and, observing that

$$V = \int_0^x X'dx' - \int_0^z Z'dz',$$

and that

$$X\delta x = Z\delta z,$$

the variation $= V'(\delta z - \delta x)$.

This result can of course be obtained at once by observing that V' is equal to the resultant vertical pressure on the solid, and that $\delta z - \delta x$ is the descent of the solid due to the ascent δx of the liquid.

EXAMPLES.

1. Prove that a homogeneous solid, in the form of a right circular cone, can float in a liquid of twice its own density with its axis horizontal, and find, in that case, the whole pressure on the surface immersed.

2. A solid formed of two co-axial right cones, of the same vertical angle, connected at the vertices, is placed with one end in contact with the horizontal base of a vessel: water is then poured into the vessel; shew that if the altitude of the upper cone be treble that of the lower, and the common density of the spindle four-sevenths that of the water, it will be upon the point of rising when the water reaches to the level of its upper end.

3. A cone, placed with its axis vertical and vertex downwards in a liquid, floats with half its axis immersed, and, when placed in another liquid, it floats with three-fourths immersed: in what proportion must these be mixed, that it may float in the mixture with two-thirds of the axis immersed?

4. A cone, of given weight and volume, floats with its vertex downwards; prove that the surface of the cone in contact with the liquid is least when its vertical angle is $2\tan^{-1}\frac{1}{\sqrt{2}}$.

5. A triangular lamina ABC, right-angled at C, is attached to a string at A, and rests with the side AC vertical and half its length immersed in liquid; prove that the density of the lamina is $\frac{7}{8}$ths of the density of the liquid.

6. A square board is placed in liquid of four times its density; shew that there are three different positions in which it will float with one given corner only below the surface of the fluid.

7. A body is floating in water; a hollow vessel is inverted over it and depressed: what effect will be produced in the position of the body, (1) with reference to the surface of the water within the vessel, (2) with reference to the surface of the fluid outside?

8. A hollow hemispherical shell has a heavy particle fixed to its rim, and floats in water with the particle just above the surface, and with the plane of the rim inclined at an angle of 45° to the surface; shew that the weight of the hemisphere : the weight of the water which it would contain

$$:: 4\sqrt{2} - 5 : 6\sqrt{2}.$$

9. A sphere of given radius floats in equilibrium in a quantity of water contained in a cylindrical vessel, revolving uniformly about its axis which is vertical; the velocity of rotation is such that the centrifugal acceleration at a distance from the axis equal to the radius of the sphere is equivalent to the acceleration of gravity; prove that the whole pressure upon the sphere varies as the cube of the surface immersed.

10. A cone of semi-vertical angle 30° and axis h floats with its axis vertical and vertex downwards in a fluid whose density is one-third greater than its own; shew that the rim of its base will be just immersed if the fluid rotate, as if rigid, with angular velocity $\sqrt{\frac{g}{h}}$ about a vertical line coinciding with the axis of the cone.

11. A solid cone is divided into two parts by a plane through its axis, and the parts are connected by a hinge at the vertex; the system being placed in water with its axis vertical and vertex downwards, shew that, if it float without separation of the parts, the length of the axis immersed is greater than $h \sin^2 \alpha$, h being the height of the cone, and 2α its vertical angle.

12. A cone, the vertex of which is fixed at the bottom of a vessel containing water, is in equilibrium, with its slant side vertical and the lowest point of its base just touching the surface. Compare the density of the cone with that of the water.

13. The curved surface of a cup is formed by the revolution of a portion of the curve $y = be^{\frac{x}{a}}$ about its asymptote. It floats in liquid with its axis vertical and narrow end downwards, and a heavier liquid is poured into it. Shew that if the cup be made of proper weight, the distance between the surfaces of the two liquids will be constant.

14. A cylinder floats in a liquid with its axis inclined at an angle $\tan^{-1}\frac{2}{5}$ to the vertical, and its upper end just above the surface; prove that the radius is $\frac{4}{7}$ of the height of the cylinder.

15. Two rods of the same substance have their ends fastened together, and float in a liquid with the angle immersed; shew that the curve of buoyancy is a parabola.

16. Find the surface of buoyancy in the case of a cone floating with its vertex immersed.

17. A hollow hemispherical cup is closed by a lid of the same small thickness and of the same substance; shew that, if it float in a liquid with its centre in the surface, the inclination of the lid to the vertical will be $11^\circ 15'$.

18. A right circular cone has a plane base in the form of an ellipse; the cone floats with its longest generating line horizontal; if 2α be the vertical angle, and β the angle between the plane base and the shortest generating line, shew that

$$\cot\beta = \cot 4\alpha - \frac{1}{5}\operatorname{cosec} 4\alpha.$$

19. If the height of a right circular cone be equal to the diameter of the base, it will float, with its slant side horizontal, in any liquid of greater density.

20. A cone, whose height is h and vertical angle 2α, has its vertex fixed at distance c beneath the surface of a liquid; shew that it will rest with its base just out of the liquid if

$$\sigma c^4 \,.\, \cos^3\alpha \,.\, \cos\theta = \rho h^4 [\cos(\theta - \alpha) \,.\, \cos(\theta + \alpha)]^{\frac{5}{2}},$$

where σ and ρ are the densities of the liquid and cone, and θ is given by the equation $c\cos\alpha = h\cos(\theta + \alpha)$.

21. A tetrahedron floats in water with one corner immersed. The three edges which meet in this corner are equal and mutually at right angles. Shew that there are one, two, or three distinct positions of equilibrium, according as the ratio of the density of the tetrahedron to that of the water is greater, equal to, or less than 4 : 27.

22. A hemispherical shell (radius $2a$) containing water rotates with an angular velocity $\sqrt{\frac{3g}{7a}}$ about its axis which is vertical: a sphere (radius a) rests on the water with its lowest point in contact with the shell without pressure on it. If the free surface passes through the rim of the shell, shew that

density of sphere : density of water :: 128 : 189.

23. An isosceles triangular lamina ABC, right-angled at C, floats with its plane vertical and the angle C immersed, in

a liquid of which the density varies as the depth; prove that, if $\frac{\pi}{4} + \theta$ be the angle which AB makes with the vertical, in either of the positions of equilibrium in which AB is not horizontal, the value of θ is given by an equation of the form

$$m \sin^2\theta \cos^2\theta = (\sin\theta + \cos\theta)^8.$$

24. A right circular cylinder, whose axis is vertical, contains a quantity of liquid, the density of which varies as the depth, and a right cone whose axis is coincident with that of the cylinder and which is of equal base, is allowed to sink slowly into the liquid with its vertex downwards. If the cone be in equilibrium when just immersed, prove that the density of the cone is equal to the initial density of the liquid at a depth equal to $\frac{1}{12}$th the length of the axis of the cone.

25. A quantity of liquid, the density of which varies as the depth, fills an inverted paraboloid, of latus rectum c, to a height h; prove that, if it be poured into a vessel of the form generated by the revolution round the axis of x of the curve,

$$a^4y^2 = 2ch^2x\,(a - x)\,(2a - x),$$

where a is any constant, its density will vary as the square of its depth.

26. A solid cone, of height h, vertical angle 2α, and density ρ, is moveable about its vertex, and its vertex is fixed at a depth c below the surface of a liquid, the density of which, at a depth z, is μz. The cone is in equilibrium with its axis inclined at an angle θ to the vertical, and its base above the surface; prove that

$$\mu c^5 \cos^8\alpha \cos\theta = 5\rho h^4 (\cos\overline{\theta + \alpha} \cos\overline{\theta - \alpha})^{\frac{5}{2}}.$$

27. A hollow paraboloidal vessel floats in water with a heavy sphere lying in it. There being an opening at the vertex, the water occupies the whole of the space between the vessel and the sphere. If the resultant pressure on the sphere be equal to half the weight of the water which would fill it, shew that the depth of the centre of the sphere below the surface of the water is $4a^2 \div 3c$, where $4a$ is the latus rectum

of the paraboloid, and c the distance of the plane of contact from the vertex.

28. A right cone floats with its vertex downwards in a fluid of which the density varies as the depth. Shew that if its axis can make an angle θ with the vertical in a position of equilibrium, then

$$5 \cos \alpha \sec \theta (\cos^2 \theta - \sin^2 \alpha)^{\frac{5}{8}} = 4 \sqrt[4]{\frac{4\sigma}{\rho}},$$

where α is the semi-vertical angle of the cone, σ its density, ρ that of the fluid at a depth equal to the slant side of the cone.

29. A right-angled triangular prism floats in a fluid of which the density varies as the depth with the right angle immersed and the edges horizontal; the curve of buoyancy is of the form

$$r^6 \sin^4 \theta \cos^4 \theta = c^6.$$

CHAPTER V.

THE STABILITY OF THE EQUILIBRIUM OF FLOATING BODIES.

63. If a floating body be slightly displaced, it will in general either tend to return to its original position, or will recede farther from that position; in the former case the equilibrium is said to be *stable*, and in the latter *unstable*, for that particular direction of displacement.

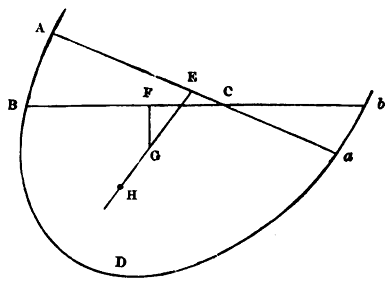
Consider first a small vertical displacement: it is clear that, if the body be floating partially immersed in homogeneous fluid, or if it be immersed, either wholly or partially, in a heterogeneous fluid of which the density increases with the depth, a depression will increase the weight of the fluid displaced, and on the contrary an elevation will diminish it; in either case the tendency of the fluid pressure is to restore the body to its position of rest, and the equilibrium is stable with regard to vertical displacements. This, it will be observed, is only shewn to be true of rigid bodies; if the increased pressure, caused by depression, have the effect of compressing any portion of the floating body, the equilibrium is not necessarily stable, and in fact it may be unstable.

An arbitrary displacement will in general involve both vertical and angular changes in the position of the body; if however the displacement be small, as we have supposed to be the case, the effects of the two changes of position can be treated independently; and we proceed to consider the effect of a small angular displacement, on the supposition that the weight of fluid displaced remains unchanged, and consequently that the fluid pressure has no tendency to raise or depress the centre of gravity of the body.

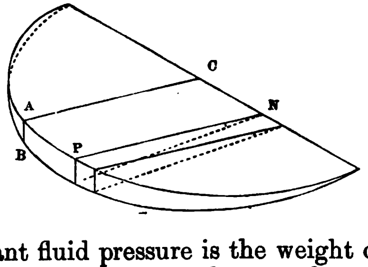
64. *A solid, floating at rest in a homogeneous liquid, is made to turn through a very small angle in a given vertical plane; to determine whether the fluid pressure will tend to restore it to its original position or not.*

Suppose the volume of liquid displaced to remain unchanged, and that the centre of buoyancy remains in the vertical plane of displacement through HG. This will be the case if CN be a principal axis of the plane of floatation.

Let AEC be the original plane of floatation and BCb the water-line after displacement through a small angle θ, G the centre of gravity of the solid, H of the fluid originally displaced, and V the volume of the fluid displaced.



In the second figure CN is the line of intersection of the two planes ACa, BCb, which is perpendicular to the plane ACB, in the first figure.



The resultant fluid pressure is the weight of $BDab$ acting upwards, and is therefore equivalent to the weight of ABa, or $g\rho V$, acting upwards through H, of the wedge aCb acting upwards, and of the wedge ACB acting downwards.

These wedges being equal, the resultant action of the two wedges is a couple, the moment of which about G is equal to its moment about C.

Taking for convenience $g\rho$ as unity, the weight of an element PN of one of the wedges

$$= \tfrac{1}{2}y^2\theta\delta x,$$

where $x = CN$, and $y = PN$; and the distance from CN of its centre of gravity is $\tfrac{2}{3}y$;

$$\therefore \text{ the moment about } CN \text{ of the wedges}$$

$$= \Sigma\left(\tfrac{1}{3}y^3\,\theta\,\delta x\right)$$

$$= \theta\textstyle\int\tfrac{1}{3}y^3\,\delta x = \theta\,.\,Ak^2,$$

where A is the area of the section ACa of the body by the plane of floatation, and k its radius of gyration relative to the line CN. Hence the restorative moment of the fluid pressure about a horizontal axis through G, parallel to CN,

$$= (Ak^2 - V\,.\,HG)\,\theta\,;$$

and if this moment is positive the solid tends to return to its original position, i.e. the equilibrium is stable

$$\text{when } HG < \frac{Ak^2}{V},$$

and conversely, is unstable

$$\text{when } HG > \frac{Ak^2}{V}.$$

If M be the point in HG through which the resultant vertical pressure of the fluid acts, in other words, if the vertical line through the centre of buoyancy meet HG in M, the moment is

$$V\,.\,GM \sin\theta,$$

$$\text{or}\quad V(HG - HM)\,\theta\,;$$

$$\therefore\; HM = \frac{Ak^2}{V},$$

and the equilibrium is stable or unstable according as $HM >$ or $< HG$.

The point M is called the *metacentre*.

If $HG = \frac{Ak^2}{V}$, that is, if M and G coincide, the equilibrium is said to be neutral.

Replacing $g\rho$, it will be seen that the restorative moment, for a displacement through a small angle θ, is

$$g\rho\theta\,(Ak^2 - V \,.\, HG).$$

65. We have assumed, in the preceding investigation, that the centre of gravity of the displaced liquid remains in the vertical plane of displacement passing through HG; when this is not the case, the expression

$$g\rho\,(Ak^2 - V \,.\, HG)\,\theta,$$

will still represent the moment of the fluid pressures, but the line of action of the resultant fluid pressure will not necessarily lie in the plane ABa.

Let $\bar{x}$ be the distance measured in the direction CN, 2nd figure, of the vertical through the centre of gravity (H') of the solid Bab, then

$$V\bar{x} = \int g\rho\theta\, xy\, dA,$$

so that $\bar{x}$ depends upon the product of inertia of the area, and vanishes when Cx and Cy are principal axes.

If the projection of the vertical through H' on the plane ABa meet HG in M, the moment of the fluid pressures about G will still be represented by $V \,.\, GM\theta$, and therefore as in the previous case $V \,.\, HM = k^2 A$, and if rotation in the direction of the plane ABa only be allowed, the position of the point M defines the stability of the equilibrium.

66. It must be observed that the above investigation is essentially statical; it is simply an inquiry into the direction in which the moment of the fluid pressure about a certain horizontal axis through G is acting in the position of displacement contemplated.

Considered dynamically, if the horizontal axis through G be not a principal axis, the forces introduced by displacement will cause accelerations about other axes through G, and will consequently produce rotations about varying axes.

Moreover a rotation about G would, except in the case in which E and C are coincident, cause a change in the quantity of fluid displaced, and vertical oscillations would therefore ensue.

67. The question of stability may be treated somewhat differently.

Defining the *metacentre* as the point of intersection with the line HG of the vertical line through the new centre of buoyancy after a slight displacement, we are led to the following theorem;

The metacentre is the centre of curvature of the curve of buoyancy at the point in the same vertical line with G.

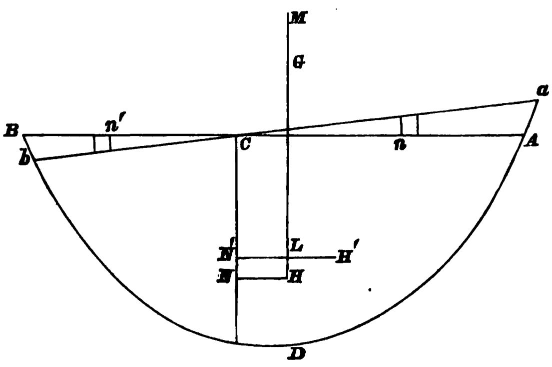
This is at once obvious from the fact that the point M is the point of intersection of consecutive normals to the curve.

Hence it appears that for any displacement, consistent with the condition that the volume displaced remains the same, the direction of the fluid pressure is always a vertical tangent to the evolute of the curve of buoyancy.

68. From the preceding theorem we can determine the expression for the height of the metacentre above the point H.

Let H be the centre of gravity of the volume ADB, and H' of aDb, aCA being a small angle θ.

Then, if α be an element of the area of the plane of floatation,



$H'N'$, HN, perpendiculars upon the vertical line through C,

$$H'N' . V - HN . V = \Sigma\,(Cn . \theta . \alpha . Cn) + \Sigma\,(Cn' . \theta . \alpha' . Cn'),$$

$$\text{or } H'L . V = \theta A k^2;$$

but if M be the centre of curvature at H,

$$H'L = H'M . \theta = HM . \theta,$$

$$\therefore\ V . HM = k^2 A.$$

The restorative moment, for a small displacement θ,

$$= g\rho V . HM . \theta = g\rho\theta\,(Ak^2 - V . HG).$$

69. The preceding article assumes that the vertical line of action of the fluid pressure, after a slight displacement, intersects HG. This will be true only when the plane of displacement is a principal section, at H, of the surface of buoyancy. When this is not the case, the projection of the line of action on the vertical plane of displacement will intersect HG in a point M, which will be the centre of curvature of the normal section of the surface.

The radius of curvature of any normal section at H, of the surface of buoyancy, is therefore $\frac{Ak^2}{V}$, and, if I and I' be the principal moments of inertia of the plane of floatation at its centre of gravity, the principal radii of curvature, at H, of the surface of buoyancy are

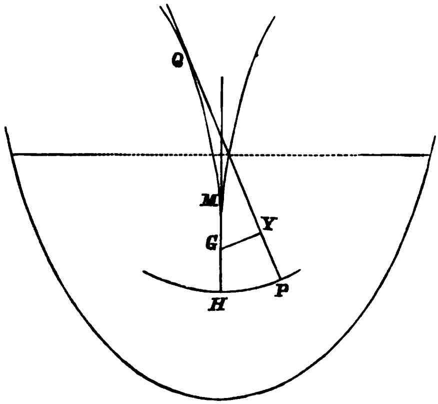
$$\frac{I}{V} \text{ and } \frac{I'}{V},$$

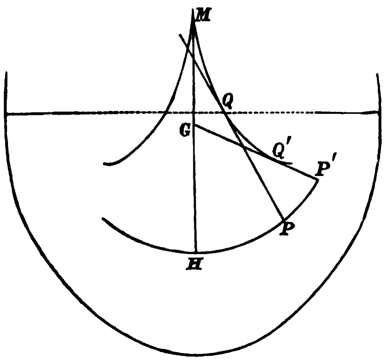
and the principal sections are parallel to the principal axes of the plane of floatation.

70. A most important case naturally presents itself; that is, the question of the stability of equilibrium of a ship when displaced by rolling.

In this case the vertical plane through HG, perpendicular to the plane of displacement, divides the floating body symmetrically, and consequently the vertical line HG passes through the point C in the plane of floatation.

The line HG also divides the curve of buoyancy symmetrically, and the point H is a point of maximum or minimum curvature. In the first of these two cases the cusp of the evolute is pointed downwards; in the second case it is pointed upwards.





The figures at once shew the effects of displacement.

In the first case the righting moment, which is the statical measure of stability for a given angle of displacement, is proportional to GY the perpendicular from G on the tangent PQ, and increases with an increase in the angle of displacement.

In the second case, the righting moment increases to a maximum value, and then diminishes, vanishing for the position given by the tangent $GQ'T'$.

This is a position of equilibrium, but it is of unstable equilibrium, in accordance with the general mechanical law that positions of stable and unstable equilibrium occur alternately.

If the equation to the curve of buoyancy be obtained in the form $p = f(\phi)$, G being the origin,

$$GY = \frac{dp}{d\phi},$$

and the righting moment is

$$W\frac{dp}{d\phi},$$

if W be the weight of the ship.

In general the curve of buoyancy, for moderate displacements, is approximately an arc of an hyperbola; in the case of a 'wall-sided' ship, that is of a ship with the sides vertical near the water-line, the curve is an arc of a parabola.

71. Taking the case of a ship floating upright, the expression for the radius of curvature of a transverse section, $2A$, of the surface of floatation is

$$r_1 = \frac{\int y^2 \tan a\, ds}{A},$$

ds being an element of the perimeter of the water-section, and a the inclination of the side of the ship to the vertical.

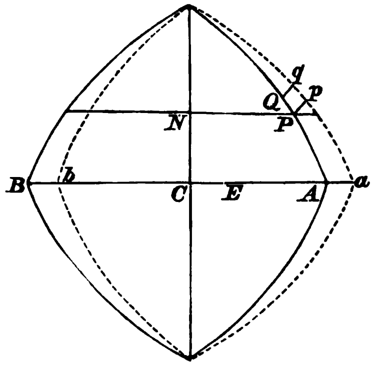
To prove this, let C' be the centre of gravity of a section through C making a small angle θ with the water-section ACB, and let aCb be the projection of the perimeter of the new section upon the water-section, E being the projection of C'.

Taking $PQ = ds$, and drawing Pp and Qq normals to the perimeter, the element of area $PQqp = y\theta \tan a ds$;

$$\therefore\; CE^2(A) = 2\int y^2\theta \tan a ds,$$

and, since $CC' = r_1\theta$, and $CE = CC'$ ultimately, it follows that

$$r_1 A = \int y^2 \tan a ds,$$



an expression first given by Mons. C. Dupin, in a memoir given to the Académie des Sciences in 1814.

A corresponding expression obviously exists for the radius of curvature (R_1) of the longitudinal section.

72. Calling r and R the metacentric heights for transversal and longitudinal displacements, that is, the radii of curvature of transverse and longitudinal sections of the surface of buoyancy; we know that

$$r = \frac{i}{V} \text{ and } R = \frac{I}{V},$$

where i and I are the principal moments of inertia of the water-section.

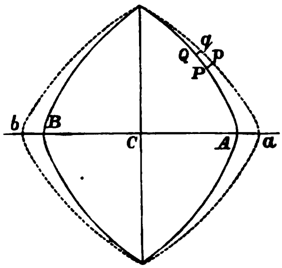
Mons. E. Leclert has established the following relations between these quantities;

$$r_1 = \frac{di}{dV} = r + \frac{Vdr}{dV}; \quad R_1 = \frac{dI}{dV} = R + V\frac{dR}{dV}.$$

A translation of Leclert's paper is given by Mr Merrifield in the *Proceedings,* for 1870, *of the Institution of Naval Architects*, and in the *Messenger of Mathematics*, March, 1872. The following is the first of the two proofs which are given.

Taking a section parallel to the water-section, and at a distance dz from it,

$$dV = Adz.$$



Let $apqb$ be the projection of this new section upon the water-section; then di is the moment of inertia of the area between ab and AB;

$$\therefore\ di = \Sigma y^2 dz \,.\, \tan\alpha\, ds,$$

$$\text{and } \frac{di}{dz} = \int y^2 \tan\alpha\, ds,$$

Hence
$$r_1 = \frac{1}{A}\frac{di}{dz} = \frac{di}{dV};$$

$$\therefore\ r_1 - r = \frac{di}{dV} - \frac{i}{V} = \frac{Vdi - idV}{VdV},$$

$$\text{or } r_1 = r + \frac{Vdr}{dV}.$$

73. We now append some examples of the determination of the metacentre.

Ex. 1. *A solid cylinder of radius a and length h floating with its axis vertical.*

In this case the plane of floatation is a circular area, and

$$Ak^2 = 4\int_0^a \tfrac{1}{3}y^3\,dx = \tfrac{4}{3}\int_0^a (a^2 - x^2)^{\frac{3}{2}}\,dx$$

$$= \tfrac{4}{3}\,a^4 \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta, \text{ putting } x = a\sin\theta,$$

$$= \frac{\pi a^4}{4};$$

therefore, if h' be the length of the axis immersed,

$$\pi a^2 h' . HM = \frac{\pi a^4}{4}, \text{ or } HM = \frac{a^2}{4h'},$$

and the equilibrium is stable if

$$\frac{a^2}{4h'} < \frac{h}{2} - \frac{h'}{2}.$$

Ex. 2. *A cylinder floating with its axis horizontal and in the surface is displaced in the vertical plane through the axis.*

The plane of floatation is a rectangle, and

$$Ak^2 = \tfrac{1}{6} a h^3,$$

h being the length of the cylinder, and a its radius;

$$\therefore HM = \tfrac{1}{3} \frac{h^2}{\pi a};$$

and the equilibrium is stable, if

$$\tfrac{1}{3} \frac{h^2}{\pi a} > \frac{4a}{3\pi},$$

$$\text{or } h > 2a.$$

Ex. 3. *A solid cone floating with its axis vertical and vertex downwards.*

Let h be the length of the axis,

z the portion of the axis immersed,

2α the vertical angle of the cone.

Then $$Ak^2 = \tfrac{1}{4}\pi z^4 \tan^4 \alpha,$$

$$\text{and } V = \tfrac{1}{3}\pi z^3 \tan^2 \alpha;$$

$$\therefore HM = \tfrac{3}{4} z \tan^2 \alpha;$$

$$\text{also } HG = \tfrac{3}{4} h - \tfrac{3}{4} z,$$

and therefore the equilibrium is stable or unstable, according as

$$z \tan^2 \alpha > \text{ or } < h - z,$$

$$\text{or } \quad z > \text{ or } < h \cos^2 \alpha.$$

But if ρ, σ, be the densities of the fluid and cone,

$$\left(\frac{z}{h}\right)^3 = \frac{\sigma}{\rho};$$

therefore the equilibrium is stable or unstable as

$$\frac{\sigma}{\rho} > \text{ or } < (\cos\alpha)^6.$$

Ex. 4. *An isosceles triangular prism floating with its base not immersed, and its edges horizontal.*

Referring to Art. 49, consider first the position of equilibrium in which the base is inclined to the horizon.

In this case, if $AQ = 2y$ and $AP = 2x$, x and y are given by the equations

$$x + y = 2a\cos^2\frac{\theta}{2},$$

$$xy = c^2.$$

The co-ordinates of G and H referred to AB, AC as axes are respectively,

$$\tfrac{2}{3}a,\quad \tfrac{2}{3}a,\quad \text{and } \tfrac{2}{3}x,\quad \tfrac{2}{3}y,$$

$$\therefore HG^2 = \tfrac{4}{9}\{(a-x)^2 + (a-y)^2 + 2(a-x)(a-y)\cos\theta\}$$

$$= \tfrac{4}{9}\{x^2+y^2+2xy\cos\theta - 2a(1+\cos\theta)(x+y) + 2a^2(1+\cos\theta)\},$$

from which, by means of the above equations, we obtain

$$HG = \tfrac{4}{3}\sin\frac{\theta}{2}\left(a^2\cos^2\frac{\theta}{2} - c^2\right)^{\frac{1}{2}}.$$

The area $PAQ = 2c^2\sin\theta$, and if M be the metacentre, and l the length of the prism,

$$2lc^2\sin\theta \,.\, HM = \frac{PQ^2}{12} \,.\, PQ \,.\, l,$$

$$HM = \frac{PQ^3}{24c^2\sin\theta}.$$

But $\qquad PQ^2 = 4(x^2 + y^2 - 2xy\cos\theta)$

$$= 16\cos^2\frac{\theta}{2}\left(a^2\cos^2\frac{\theta}{2} - c^2\right);$$

$$\therefore HM = \tfrac{4}{3}\frac{\cos^2\frac{\theta}{2}}{c^2\sin\frac{\theta}{2}}\left(a^2\cos^2\frac{\theta}{2} - c^2\right)^{\frac{3}{2}},$$

$$\text{and } HM > HG, \text{ if } c^2\sin^2\frac{\theta}{2} < \cos^2\frac{\theta}{2}\left(a^2\cos^2\frac{\theta}{2} - c^2\right),$$

$$\text{i.e. if } \cos^2\frac{\theta}{2} > \frac{c}{a}.$$

Next, consider the case in which the base is horizontal, and PQ therefore parallel to BC.

The area $PAQ = 2c^2 \sin\theta$,

$$AP = AQ = 2c, \quad \text{and } PQ = 4c\sin\frac{\theta}{2}.$$

$$\text{Hence, } HM = \tfrac{4}{3}c\frac{\sin^2\frac{\theta}{2}}{\cos\frac{\theta}{2}}, \text{ and } HG = \tfrac{4}{3}(a-c)\cos\frac{\theta}{2},$$

$$\text{and } HM > HG \text{ if } \cos^2\frac{\theta}{2} < \frac{c}{a}.$$

Now in the Art. 49, before referred to, we have shewn that there are three positions of equilibrium, or one only, according as

$$\cos^2\frac{\theta}{2} > \text{or} < \frac{c}{a}.$$

Hence it follows, that when there are three positions of equilibrium, the intermediate one, in which CB is horizontal, is a position of unstable equilibrium, while in the other two positions the equilibrium is stable.

If there be only one position in which the prism will rest, its equilibrium is stable.

It will be a useful exercise for the student to obtain these results by investigating the equation to the curve of buoyancy, and determining the position of its centre of curvature.

74. *Finite displacements.* If a solid body, floating in water, be turned through any given angle from its position of equilibrium, then, as before, the moment of the fluid pressure is restorative or not according as the point L at which

the vertical through the new centre of buoyancy meets the line HG is above or below G.

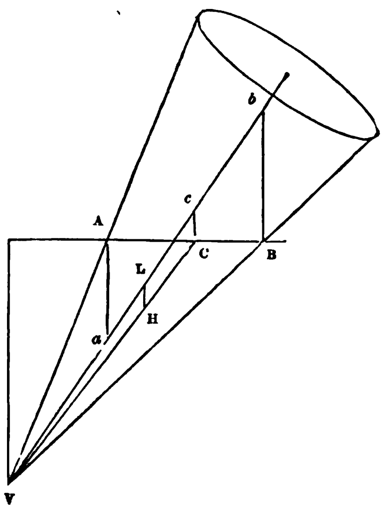
It is not to be inferred that if L is above G, the body will when set free return to its original position and oscillate through it, or even that the original position is one of stable equilibrium, according to our previous definition of stability: it is a general law of mechanics that positions of stable and unstable equilibrium occur alternately, and the body may have been displaced from its original position *through* other positions of equilibrium.

As a particular example take the following.

A solid cone, floating with its axis vertical and vertex downwards, is turned through an angle θ in a vertical plane, the volume of fluid displaced remaining the same; to determine the direction of the moment of the fluid pressure.

Let AB be the major axis of the elliptic section made by the surface plane of the fluid, C its middle point, Aa, Bb, Cc, lines at right angles to AB, and let the angle $AVB = 2\alpha$ and $VA = d$. Then

$$VAa = \theta - \alpha, \text{ and } VBb = \pi - \theta - \alpha.$$



$$Vc = \tfrac{1}{2}(Va + Vb) = \tfrac{1}{2} \cdot \left\{d\frac{\sin(\theta-\alpha)}{\sin\theta} + d\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\frac{\sin(\theta+\alpha)}{\sin\theta}\right\}$$

$$= \frac{d\cos\theta}{\cos(\theta+\alpha)};$$

$$\therefore\ VL = \tfrac{3}{4}\,d\frac{\cos\theta}{\cos(\theta+\alpha)}.$$

The semi-minor axis of the ellipse AB is a mean proportional between the perpendiculars from A and B on the axis of the cone,

$$\therefore\ \text{its area} = \pi\,\tfrac{1}{2}\,AB\,(VA\,.\,VB\,.\,\sin^2\alpha)^{\frac{1}{2}}$$

$$= \frac{\pi}{2}d^2\frac{\sin\alpha\sin 2\alpha}{\cos(\theta+\alpha)} \cdot \left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{1}{2}};$$

therefore the volume of the fluid displaced

$$= \tfrac{1}{3}\,d\cos(\theta-\alpha)\,.\,(\text{area of ellipse})$$

$$= \tfrac{1}{3}\,\pi d^3\sin^2\alpha\cos\alpha\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{3}{2}}.$$

Hence, if ρ, σ, be the densities of the fluid and the cone, since the weight of the fluid displaced is equal to that of the cone, we have

$$\rho d^3\sin^2\alpha\cos\alpha\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{3}{2}} = \sigma h^3\tan^2\alpha,$$

$$\text{or}\ \left(\frac{d}{h}\right)^3 = \frac{\sigma}{\rho}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}}\frac{1}{\cos^3\alpha}.$$

And $VL > VG$ if

$$d\frac{\cos\theta}{\cos(\theta+\alpha)} > h,$$

$$\text{or if}\qquad \sqrt[3]{\frac{\sigma}{\rho}} > \frac{\cos\alpha\cos(\theta+\alpha)}{\cos\theta}\cdot\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{1}{2}}.$$

Supposing θ indefinitely small, we obtain the condition of stability for an infinitesimal displacement,

$$\sqrt[3]{\frac{\sigma}{\rho}} > \cos^2\alpha;\ \text{as before, Ex. 3, Art. 73.}$$

Let the equilibrium of the cone be neutral, that is, let

$$\sigma = \rho \cos^6 \alpha,$$

then, after a finite displacement, the action of the fluid will tend to restore the cone to its original position, if

$$\cos \alpha \, . \cos \theta > \sqrt{\{\cos (\theta + \alpha) \, . \cos (\theta - \alpha)\}},$$

a condition which is always true, α and θ being each less than a right angle.

In the case of neutral equilibrium of a cone, the equilibrium may therefore be characterised as stable for any finite displacement.

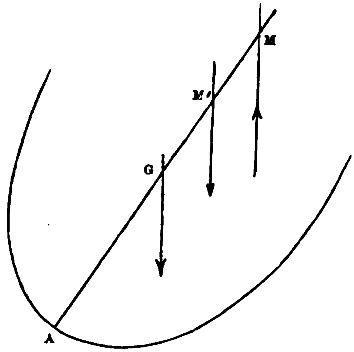
75. When liquid is contained in a vessel, which is slightly displaced from its original position, the preceding investigations enable us to determine the line of action of the resultant *downward* pressure.

The problem in fact in this case, as in the previous case, is the following.

A given volume, the centre of gravity of which is H, is cut from a solid ABC by a plane, and the line CH is perpendicular to the plane; the same volume being cut off by a plane making a very small angle with the plane AB, to determine the position of the straight line perpendicular to the second plane, and passing through the centre of gravity of the volume cut off by it.

If the interior surface of the vessel is symmetrical with respect to the plane through H perpendicular to the line of intersection of the two planes, the line whose position is required will intersect CH in a point M, *the metacentre*, the position of which is determined by our previous results.

76. *A hollow vessel containing liquid, floats in liquid; required to determine the nature of the equilibrium, supposing that the body is symmetrical with respect to the vertical plane of displacement through its centre of gravity, and that the centres of gravity of the body and of the liquid are in the same vertical line.*



Let M be the metacentre for the displaced fluid, and M' for the contained fluid, W, W', the weights of the displaced and contained fluid*.

Taking moments about G, the centre of gravity of the vessel, the resultant fluid pressures will tend to restore equilibrium, or the reverse, according as

$$W \,.\, GM - W' \,.\, GM'$$

is positive or negative, i.e. as

$$\frac{W}{W'} > \text{ or } < \frac{GM'}{GM}.$$

Ex. *A hollow cone containing water floats in water with its axis vertical.*

Let h = the length of the axis of the cone,

h' = the length of the axis in the contained fluid,

z = the length beneath the surface of the external fluid.

Taking 2α as the vertical angle of the cone, we have

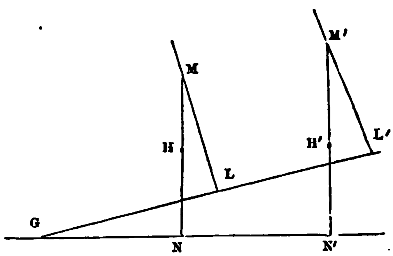
$$HM = \tfrac{3}{4}\, z \tan^2 \alpha.$$

* This is the case of a leaky ship rolling; the next article discusses the pitching of a leaky ship.

But
$$HG = \tfrac{2}{3}h - \tfrac{3}{4}z;$$
$$\therefore GM = \tfrac{3}{4}z\sec^2\alpha - \tfrac{2}{3}h.$$
Similarly
$$GM' = \tfrac{3}{4}h'\sec^2\alpha - \tfrac{2}{3}h,$$
also
$$\frac{W}{W'} = \frac{z^3}{h'^3};$$
therefore the equilibrium is stable if
$$\left(\frac{z}{h'}\right)^3 > \frac{9h'\sec^2\alpha - 8h}{9z\sec^2\alpha - 8h},$$
z being given by the equation
$$W - W' = \tfrac{1}{3}g\rho\pi\tan^2\alpha\,(z^3 - h'^3) = \text{weight of cone}.$$

77. In the case in which the centres of gravity of the contained and of the fluid displaced are not in the same vertical, suppose the displacement to take place in direction of the vertical plane through the centres of gravity, and that the body is symmetrical with respect to that plane.

Let G be the centre of gravity of the body, H of the fluid displaced, H' of the contained fluid, and M, M', the metacentres.



Also let GNN' be horizontal in the position of equilibrium, and GLL' the horizontal line through G in the displaced position.

Then W, W', having the same meanings as before, and θ being the angle of displacement, the equilibrium is stable or unstable, as

$$W \,.\, GL > \text{or} < W' \,.\, GL',$$

or

$$W(GN\cos\theta + MN\sin\theta) > \text{or} < W'(GN'\cos\theta + M'N'\sin\theta),$$

$$\text{i.e. since } W \,.\, GN = W' \,.\, GN',$$

as
$$\frac{W}{W'} > \text{or} < \frac{M'N'}{MN}.$$

78. *Stability of the equilibrium of bodies floating under constraint.*

In those cases of constraint, in which, for a small displacement, the volume of liquid displaced remains unchanged, the theory of the metacentre determines the line of action of the fluid pressure, and the question of stability is then easily determined.

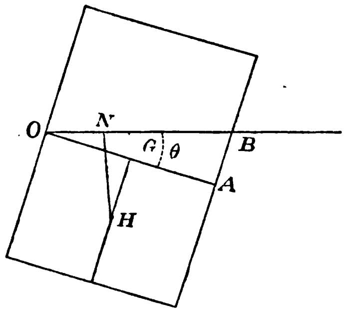
Suppose, for instance, that a body, partially immersed, is moveable about a horizontal axis, which is vertically beneath the centre of gravity (C) of the plane of section of the body by the surface of the liquid.

The effect of a displacement through a small angle θ will be to depress the point C through a space which depends upon θ^2, and therefore, to the first order of small quantities, the volume displaced remains unchanged, and the metacentre is the same as if C remained in the surface.

If the body be moveable about a horizontal axis which is not vertically beneath the point C, the change in the volume displaced cannot be neglected, and the question of stability must be treated by a direct consideration of the action of the displaced liquid.

Ex. *A rectangular lamina rests in a liquid of twice its own density with two of its sides vertical, and is moveable in its own plane about the middle point of one of its vertical sides.*

The figure represents the lamina when slightly displaced through an angle AOB, (θ), the point O which is in the surface being the middle point.



Then if $OA = a$, and if the height $= 2b$, the area

$$AOB = \tfrac{1}{2}\, a^2\theta,$$

and, taking moments about O, the equilibrium is stable if

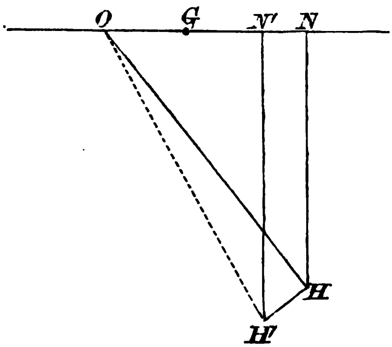
$$2\rho\,(\tfrac{1}{2}\, a^2\theta \,.\, \tfrac{2}{3}\, a + ab \,.\, ON) > \rho \,.\, 2ab \,.\, \frac{a}{2},$$

HN being the vertical through H;

$$\text{or, since } ON = OG \cos\theta - HG \sin\theta = \frac{a}{2} - \frac{b}{2}\,\theta,$$

$$\text{if, } 2a^2 > 3b^2.$$

79. In the particular case in which the centre of gravity of the body and the axis about which it is moveable are in the surface of the liquid, a formula can be given, for the determination of stability, analogous to that of Art. 64.



Taking Oy as the axis, and V as the volume of displaced liquid.

Loss of moment due to the displacement of H

$$= g\rho V . NN' = g\rho V . HN . \theta,$$

and restorative moment due to the increase of the displaced liquid

$$= \int g\rho \tfrac{1}{2} x^2\theta . \tfrac{2}{3} x dy = g\rho \, Ak^2\theta,$$

Ak^2 being the moment of inertia about Oy of the area of the plane of floatation;

$$\therefore \text{ the equilibrium is stable, if } Ak^2 > V . HN.$$

In the general case of a horizontal axis, if O', L, and N be the projections, on the plane of floatation, of O, G, and H, the test of stability is that

$$g\rho Ak^2 - g\rho V . HN + W . GL$$

must be positive, with the condition,

$$W . O'L = g\rho V . O'N,$$

Ak^2 being the moment of inertia, about $O'y$, of the area of the plane of floatation.

80. *The equilibrium of bodies floating in two liquids.*

Suppose the body to be wholly immersed with the portion V of its volume in the upper liquid and V' in the lower.

Take the case in which the centres of gravity H, H', of the liquids displaced by V and V', and therefore the centre of gravity G of the body, are in the same vertical.

Displace the body through a small angle θ about an axis through C the centre of gravity of the section ACB of the body by the common surface of the liquids.

The action then consists of a pressure $g\rho' V'$ acting upwards through M' the metacentre for the lower liquid, and of a presure $g\rho V$ acting upwards through M the metacentre for the upper liquid.

If M be above G and M' below G, the equilibrium is stable, if

$$g\rho V.GM > g\rho' V'.GM',$$

or, observing that HM is measured downwards from H,

$$\text{if} \quad \rho V.HG - \rho Ak^2 > \rho' V'.H'G - \rho' Ak^2,$$

$$\text{or} \quad (\rho' - \rho) Ak^2 > \rho' V'.H'G - \rho V.HG,$$

Ak^2 being the moment of inertia of the area ACB about the axis through C.

If K be the centre of gravity of the fluid displaced, L the resultant metacentre, and W the weight of the body,

$$\begin{aligned} W.KL &= g\rho V.KM - g\rho' V'.KM', \\ &= g\rho V(KH - HM) - g\rho' V'(KH' - H'M') \\ &= g(\rho' - \rho) AK^2, \end{aligned}$$

and the equilibrium is stable, if L is above G.

81. The preceding question may be also usefully treated in the following manner.

The body may be supposed to be completely immersed in a liquid of density ρ, and we can then imagine a liquid of density $\rho' - \rho$ superposed.

Let E be the centre of gravity of the whole volume $V + V'$, so that $(V + V').EG = V'.H'G - V.HG$.

If the body be displaced through a small angle θ, the restorative moment is

$$g(\rho' - \rho)(Ak^2 - V'.H'G) - g\rho(V + V').EG,$$

which by the above relation becomes

$$g(\rho' - \rho) Ak^2 - g\rho' V'.H'G + g\rho V.HG,$$

and the stability depends upon the sign of this expression.

82. *Stability of a body floating in heterogeneous liquid.*

We shall consider only the case in which the body is symmetrical with regard to the line HG so that this line

passes through C, the centre of gravity of the water-section, and contains the centre of gravity of all the strata of liquid displaced.

The effect of this limitation is that a small displacement (θ) about C, or about any point in HG, raises the centre of gravity of any horizontal section through a height of the order θ^2, and we can therefore employ the formulæ of Arts. (64) and (80).

83. *Determination of the metacentre for a body floating in heterogeneous liquid.*

A liquid in which the density is a function of the depth, can be conceived as made up of a series of homogeneous liquids having successive descending surfaces, and the centre of gravity H of the whole mass U displaced, will be the centre of gravity of the aggregate of these liquids.

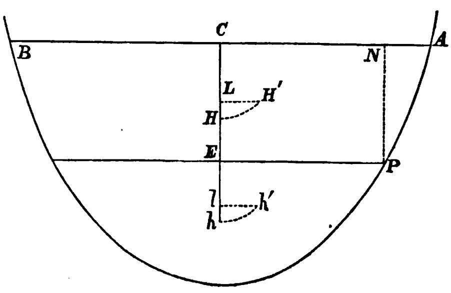
Turning the body through a small angle θ, the vertical tilt of the centre of gravity of each portion will vary as θ^2, and therefore the vertical tilt of H will vary as θ^2.

If E be the surface, ρ the density, and V the volume displaced of one of these liquids, and if h be the centre of gravity of V, then, as in Art. (68),

$$h'l \, . \, V = \theta A k^2,$$

A being the area of the section E;

therefore $$H'L \, . \, U = \Sigma h'l \, . \, V\rho = \theta \Sigma \rho A k^2.$$



Further, if M be the metacentre,

$$H'L = H'M \, . \, \theta = HM \, . \, \theta,$$

and therefore $$HM \, . \, U = \Sigma \rho A k^2.$$

This formula includes the case in which the solid bulges out below the water-section.

Taking CA as the axis of y, and considering the case when the solid does not bulge out,

$$\Sigma \rho A k^2 = \iint \Sigma(\rho) \, y^2 \, dxdy,$$

the double integration extending over the water-section, and the summation of ρ down the vertical ordinate NP; hence, if ρ' be the density at P,

$$HM \, . \, U = \iint \rho' y^2 \, dxdy.$$

84. If the floating body be a solid of revolution, having its axis vertical, the formulæ can be somewhat simplified.

For, transferring to polar co-ordinates,

$$HM \, . \, U = 4 \int_0^{\frac{\pi}{2}} \int_0^a \rho' r^3 \sin^2 \theta dr \, d\theta$$

$$= \int_0^a \pi \rho' r^3 \, dr,$$

ρ' being the density corresponding to the section of radius r.

Suppose, for example, that the density varies as the depth, and that the floating body is a cone, vertex downwards.

If h be the length of axis immersed,

$$U \, . \, HM = \int_0^{h \tan \alpha} \pi \mu \, (h - r \cos \alpha) \, r^3 dr = \frac{\pi \mu h^5 \tan^4 \alpha}{20},$$

and $$U = \int_0^h \pi \mu z \, (h - z)^2 \tan^2 \alpha dz = \frac{\pi \mu h^4 \tan^2 \alpha}{12},$$

$$\therefore \; HM = \tfrac{3}{5} h \tan^2 \alpha.$$

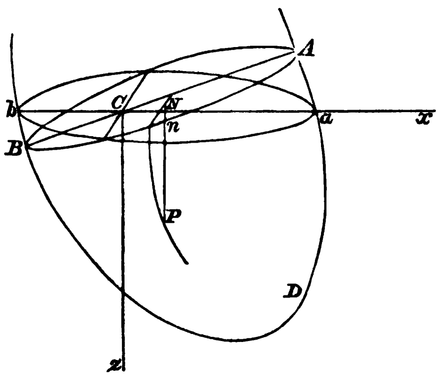
Also, if V be the vertex,

$$VH = \tfrac{3}{5}h;$$

$$\therefore\ VM = \tfrac{3}{5}h \sec^2 \alpha.$$

85. The following is another method of determining the stability or the instability of the equilibrium of a floating body*.

Taking the case in which the body is symmetrical with regard to the vertical plane zx of displacement, let ACB be the plane of floatation, C being its centre of gravity, and aCb the liquid-surface after the body has been turned about Cy through a small angle θ, Cx being horizontal and Cz vertical.



Let x, y, z be co-ordinates of a point P in the surface of the body, and let the vertical ordinate Pn meet the plane xy in n, and the plane ACB in N, then

$$Nn = x\theta, \text{ and } PN = z + x\theta.$$

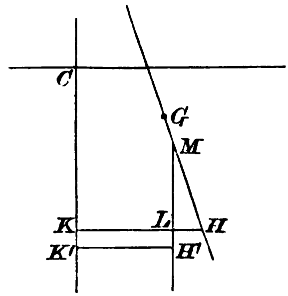
Let H' be the centre of gravity of aDb, H of ADB, and G of the whole body.

* Arts. (85—88) were originally published in the Second Edition (1867) of this Treatise.

Then, V being the volume of liquid displaced, and HK, $H'K'$ perpendiculars on Cz,

$$V.H'K' = \iint xz\,dy\,dx, \text{ and } V.HK = \iint x(z + x\theta)\,dy\,dx;$$

$$\therefore\ V.HL = \iint x^2\theta\,dy\,dx.$$



But, if the vertical through H' meet HG in M, the metacentre,

$$HL = HM.\theta;$$

$$\therefore\ V.HM = \iint x^2\,dy\,dx,$$

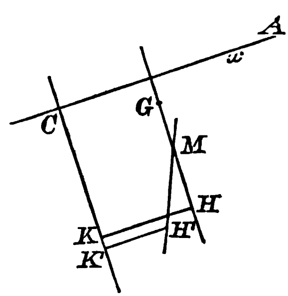
and if HM be greater than HG, the equilibrium is stable.

The expression $\iint x^2\,dy\,dx$, is the moment of inertia of the plane area aCb about Cy, and is ultimately the same as that of ACB about Cy.

Hence, if A be the area ACB, the equation may be written

$$V.HM = k^2A.$$

86. This result may also be obtained by taking CA as axis of x, but the process is then somewhat longer, as it becomes necessary to shew that, to the first order of small quantities,



$$CK = CK'.$$

This however is easily seen, for

$$V.CK = \iint \frac{z}{2}.z\,dy\,dx,$$

and $$V.CK' = \iint \tfrac{1}{2}(z + x\theta)(z - x\theta)\,dy\,dx,$$

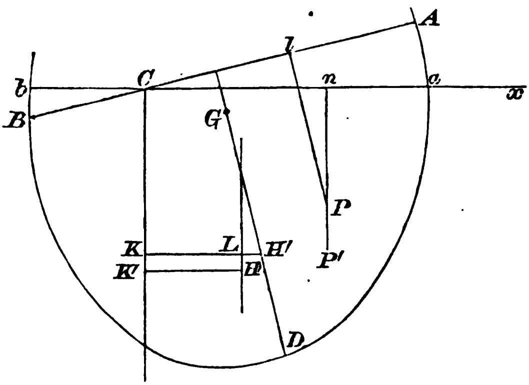
and therefore $V.KK' = \iint \frac{1}{2} x^2\theta^2 \, dy \, dx,$

which is of the second order.

Again, we may give another form to the process by a direct investigation of the moment, about Cy, of the fluid pressure.

87. *A body is floating in equilibrium in a fluid, of which the density is, at any point, a function of the depth of that point; it is required to find the condition of stability.*

Taking ACB as the plane of floatation, and aCB as the liquid-surface after displacement, let H' be the centre of gravity of the fluid displaced by aDb.



Take the vertical through C as axis of z, and Ca as axis of x; then if P be any point (x, y, z) within the body, and Pl the perpendicular on CA,

$$Pl = z\cos\theta + x\sin\theta$$
$$= z + x\theta \text{ ultimately.}$$

And if the density $\rho = f$ (the depth), the density at P before displacement

$$= f(z + x\theta)$$
$$= f(z) + x\theta f'(z),$$

to the first order.

Hence, if U be the mass of liquid displaced,

$$U\,.\,HK=\iiint_{-x\theta}^{\,'}\{f(z)+x\theta f'(z)\}\,x\,dz\,dy\,dx,$$

$$U\,.\,H'K'=\iiint_{0}^{z'}f(z)\,x\,dz\,dy\,dx,$$

where z' is the length, nP', of the ordinate nP produced to meet the surface of the body in P'.

By subtraction, we obtain

$$U\,.\,HL=\iiint_{-x\theta}^{0}f(z)\,x\,dz\,dy\,dx+\iiint_{-x\theta}^{z'}\theta x^2 f'(z)\,dz\,dy\,dx.$$

In the first integral z is less than $x\theta$, and therefore $f(z)$ to the first order is equal to

$$f(0)+zf'(0);$$

the first integral then

$$=f(0)\iint x^2\,\theta\,dy\,dx \text{ to the first order,}$$

and the second integral

$$=\theta\iint\{f(z')-f(-x\theta)\}\,x^2\,dy\,dx.$$

Further, $\qquad f(-x\theta)=f(0)-x\theta f'(0),$

and therefore $\qquad U\,.\,HL=\theta\iint x^2 f(z')\,dy\,dx,$

neglecting small quantities of the second order.

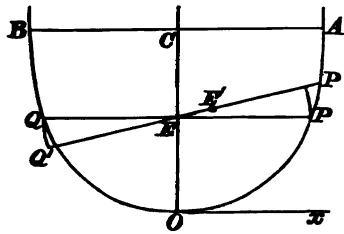
But $HL=\theta\,.\,HM$, and we thus obtain

$$U\,.\,HM=\iint x^2 f(z')\,dy\,dx,$$

the integration extending over the plane of floatation.

This result can also be obtained, as in the previous case, by taking CA as the axis of x, or by directly investigating the moment of the fluid pressure about Cy.

88. For the particular case of a solid of revolution, the following proof may be given.



Take the vertex O as origin, the axis of the body for axis of z, and Ox horizontal. Let PEQ be a horizontal section, and $P'EQ'$ a section through E inclined at a small angle θ to PEQ.

Then if ϕ be the angle between EP and the tangent at P,

$$EP' = EP + EP \cdot \theta \cot \phi,$$

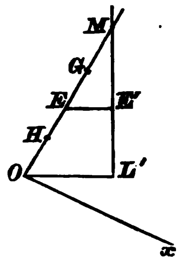
and $$EQ' = EQ - EQ \cdot \theta \cot \phi, \text{ ultimately};$$

$$\therefore P'Q' = PQ \text{ to the 1st order,}$$

and the area $P'EQ'$ = that of PEQ.

If E' be the middle point of $P'Q'$,

$$EE' = \tfrac{1}{2}(EP' - EQ') = EP \cdot \theta \cdot \cot \phi = x\theta \cot \phi.$$



In the position of displacement, let OL' be the perpendicular from O upon the vertical through E';

then $$OL' = OE \cdot \theta + EE'$$
$$= z\theta + x\theta \cot \phi,$$

and the moment about O of the fluid pressure

$$= \Sigma g\rho\pi x^2 dz \,\{z\theta + x\theta \cot \phi\}$$

$$= \int_0^c g\theta\rho\pi x^2 \left\{z + x\frac{dx}{dz}\right\} dz, \text{ where } OC = c.$$

If the equilibrium be stable, this moment must be greater than

$$W \cdot OG \cdot \theta,$$

W being the weight of the body.

Also $$W \, . \, OH = \int g\rho\pi x^2 z dz,$$

and, if the line of action of the resultant fluid pressure meet the axis in M, the moment $= W \, . \, OM \, . \, \theta$;

$$\therefore W \, . \, OM = \int g\rho\pi x^2 \left(z + x\frac{dx}{dz}\right) dz;$$

and $$W \, . \, HM = \int_0^c g\rho\pi x^3 \frac{dx}{dz} dz,$$

$$= \int_0^a g\rho\pi x^3 dx, \text{ if } CA = a,$$

remembering that ρ is a function of z and therefore of x.

EXAMPLES.

1. An inverted vessel formed of a substance which is heavier than water contains enough air to make it float; prove that, if it be pushed down through a certain space, it will be in a position of equilibrium which for vertical displacement will be unstable.

2. A solid cylinder, one end of which is rounded off in the form of a hemisphere, floats with the spherical surface partly immersed: find the greatest height of the cylinder which is consistent with stability of equilibrium.

3. If a solid paraboloid, bounded by a plane perpendicular to its axis, float with its axis vertical and vertex immersed, the height of the metacentre above the centre of gravity of the displaced liquid is equal to half the latus rectum.

4. A cone, whose vertical angle is 60°, floats in water with its axis vertical and vertex downwards; shew that its metacentre lies in the plane of floatation; and that its equilibrium will be stable provided its specific gravity $> \frac{27}{64}$.

5. An isosceles wedge floats with its base horizontal, and its edge immersed; shew that the equilibrium is stable for displacements in a plane perpendicular to the edge, if the ratio of the density of the wedge to that of the fluid is greater than the ratio $(\cos \alpha)^4 : 1$, 2α being the angle of the wedge.

6. A closed cylindrical vessel, quarter-filled with ice, is placed floating in water with its axis vertical; the weight of the vessel is one-fourth of the weight of the water which it can contain; examine the nature of the equilibrium before and after the ice melts, neglecting the change of volume consequent on the change of temperature.

7. Find a solid of revolution such that, when a segment of it is immersed in liquid, the distance between the centre of buoyancy and the metacentre may be constant, whatever be the height of the segment.

8. Water rests upon mercury, and a cone is too heavy to rest without its vertex penetrating the mercury; find the density of the cone that the equilibrium may be stable.

9. If the floating solid be a cylinder, with its axis vertical, the ratio of whose specific gravity to that of the fluid is σ, prove that the equilibrium will be stable, if the ratio of the radius of the base to the height be greater than $\{2\sigma(1-\sigma)\}^{\frac{1}{2}}$.

10. A vessel in the form of a paraboloid of revolution contains water, and rests with its vertex on the highest point of a fixed rough sphere; find the condition that the equilibrium may be stable.

11. If a cylindrical shell without weight contain liquid and float in another liquid, shew that the equilibrium will be stable, unless the ratio of the density of the internal to the external fluid is less than unity, and greater than half the duplicate ratio of the radius of the cylinder to the depth of the internal fluid.

12. A hemispherical shell, containing liquid, is placed on the vertex of a fixed rough sphere of twice its diameter; prove that the equilibrium will be stable or unstable, as the weight of the shell is greater or less than twice the weight of the liquid.

13. A solid of revolution floats with its vertex downwards, determine its form when the position of the metacentre is independent of the density of the liquid.

14. A conical shell, vertex downwards, floats in unstable equilibrium; how much water must be poured in to make the equilibrium stable?

15. A solid cone is placed in a liquid with its axis vertical, and with its vertex downwards and resting on the base of the vessel containing the liquid. If the depth of the liquid be half the height of the cone, and its density four times the density of the cone, prove that the equilibrium will be stable if the vertical angle of the cone exceeds 120°.

Replacing the solid cone by a thin conical shell of the same height, of vertical angle 60°, containing liquid, up to the level of the middle point of its axis, of half the density of the liquid outside, prove that the equilibrium will be stable if the weight of the shell be less than three-fourths of the weight of the liquid inside.

16. A cylindrical vessel, the weight of which may be neglected, contains water, and the vessel is placed on the vertex of a fixed rough sphere with the centre of its base in contact with the sphere. Find the condition of stability for infinitesimal displacements, and prove that, if the equilibrium be neutral for such displacements, it will be unstable for small finite displacements.

17. Find the form of a solid of revolution floating with its axis vertical, and such that the distances of the metacentre and the centre of buoyancy from the lowest end of the solid may be in a constant ratio whatever be the density of the liquid.

18. A semicircular cylinder rests with its axis vertical in a liquid of twice its own density; if it be moveable about the line of intersection of its vertical plane face with the surface, find the condition of stability.

19. A right circular cone floats with its axis horizontal in a liquid the density of which is double that of the cone, the vertex being attached to a fixed point in the surface

of the liquid; prove that for stability the vertical angle must be less than 120°.

20. A cylindrical vessel is moveable about a horizontal axis passing through its centre of gravity, and is placed so as to have its axis vertical; if water be poured in, shew that the equilibrium is at first unstable; and find the condition which must be satisfied, in order that it may be possible to make the equilibrium stable by pouring in enough water.

21. A thin conical vessel of given weight is moveable about a diameter of its base, which is horizontal, and is partly filled with a heavy fluid; shew that the equilibrium is always stable if the semivertical angle of the cone is $< 30^\circ$; and if it be greater than this, determine when the equilibrium is stable or unstable.

22. Water is contained in a vessel having a horizontal base, and a paraboloid whose specific gravity is four-ninths that of water, and the length of whose axis is to the latus rectum as nine to eight, is supported partly by the fluid and partly by the base on which the vertex rests; find the least depth of the fluid for which the equilibrium is stable.

23. A parabolical cup, the weight of which is W, standing on a horizontal table, contains a quantity of water, the weight of which is nW; if h be the height of the centre of gravity of the cup and the contained water, the equilibrium will be stable provided the latus rectum of the parabola be

$$> 2(n+1)h.$$

24. A solid of revolution floats with its axis vertical, and is sunk to different depths by placing weights at a fixed point on its axis.

Find the form of the solid that the equilibrium may always be neutral.

25. A solid cone whose axis is vertical and vertex downwards is moveable about an axis coincident with a generating line; to what depth must the system be immersed in water, in order that the equilibrium of the cone may be stable?

26. A solid of cork bounded by the surface generated by the revolution of a quadrant of an ellipse about the axis major sinks in mercury up to the focus. If the equilibrium be neutral for small angular displacements, prove that

$$2e^4 + 4e^3 + 2e^2 - e - 2 = 0.$$

27. The solid formed by a portion of $cy^2 = z(a^2 - x^2)$ cut off by a plane parallel to that of xy floats in a fluid of n times its density; prove that, if it is in neutral equilibrium for small angular displacements in any vertical plane,

$$n^{\frac{2}{3}} = 1 + \tfrac{5}{8}\frac{a^2}{c^2}.$$

28. An isosceles triangular lamina ABC floats with its base AB horizontal, and above the surface, in a liquid, the density of which varies as the depth: if h be the depth of C below the surface, the height of the metacentre above C is

$$\tfrac{1}{2} h \sec^2 \frac{C}{2}.$$

29. An elliptic lamina floats half immersed, with its transverse axis ($2a$) vertical, in a liquid, the density of which varies as the square of the depth; prove that the depth of the metacentre is

$$\tfrac{32}{15}\frac{ae^2}{\pi},$$

e being the eccentricity.

30. A right circular cylinder rests in a liquid with its axis vertical and a length c immersed. The density at a depth z being $\phi(z)$, shew that the depth of the metacentre is

$$\frac{\int_0^a z\phi(z)\,dz - \frac{a^2}{4}\phi(a)}{\int_0^a \phi(z)\,dz}.$$

31. A paraboloid of revolution floats with its axis vertical and vertex downwards in a liquid, the density of which varies as the depth; the equilibrium will be stable or

unstable, according as $4c$ is less or greater than $3(m+a)$, where c is the length of the axis, a the length immersed, and m the latus rectum of the generating parabola.

32. A prolate spheroid floats half immersed, with its axis vertical, in a liquid, the density of which varies as the square of the depth; the height of the metacentre above the surface is

$$\frac{5}{8}\frac{a^2-b^2}{b}.$$

33. A solid paraboloid of revolution floats with its axis vertical, vertex downwards, and focus in the surface of a liquid, the density of which at the depth z is $\mu(a+z)$, $4a$ being the latus rectum of the generating parabola; prove that the distance of the metacentre from the vertex is $\frac{21}{8}a$.

34. A homogeneous cone floats with its vertex downwards in a liquid whose density varies as the square of the depth; if the density of the cone be equal to that of the liquid at a depth equal to a fifth of the height of the cone, the vertical angle, when the equilibrium is neutral, is given by the equation,

$$\cos^2\alpha = \frac{2}{3}\left(\frac{2}{5}\right)^{\frac{1}{5}}.$$

35. A solid paraboloid of height h and latus rectum $4a$, is in equilibrium in a vertical position, with its vertex downwards, and is moveable about its vertex, which is fixed at a given depth c below the surface of a liquid, the density of which varies as the depth; prove that the equilibrium is stable if the ratio of the density of the paraboloid to the density of the liquid at the depth of its vertex is less than the ratio of c^3+4ac^2 to $4h^3$.

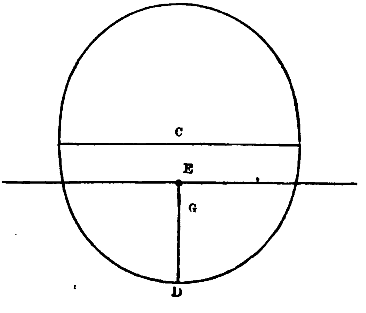
CHAPTER VI.

OSCILLATIONS OF FLOATING BODIES.

89. A HEAVY body which is floating in liquid in a position of stable equilibrium, will, if slightly displaced from that position, make small vertical and angular oscillations; we proceed to consider, in a simple case, the laws of these oscillations. We shall suppose that the body is symmetrical with regard to a vertical plane through its centre, and that the initial displacement is parallel to this plane.

It is evident that the subsequent motions of all points of the body will be parallel to this plane, and if the equilibrium be stable, that the motion will consist of small vertical and angular oscillations.

First, let the vertical line through G and H (CED) pass through the centre of gravity of the plane of floatation. When this is the case we can consider the vertical and angular displacements independently of each other.



Suppose a small vertical displacement; then the portion CE of the body which is raised out of the fluid may be considered as a thin cylinder.

Let $CE = z$, then $EG = CG - z$, and

the moving force downwards on the body = the weight of the body − the weight of the fluid displaced

$$= g\rho A \,.\, z,$$

if A be the area of the plane of floatation;

$$\therefore\ m \frac{d^2 \,.\, EG}{dt^2} = g\rho A z,$$

m being the mass of the body.

But mg = the weight of fluid displaced

$= g\rho V$, V being the volume CD;

$$\therefore\ \frac{d^2 z}{dt^2} + \frac{gA}{V} z = 0,$$

is the equation which determines the motion.

The time of a complete oscillation is therefore

$$2\pi \sqrt{\left(\frac{V}{gA}\right)}.$$

90. Next suppose a small angular displacement (α) about C, then G is raised through a space which depends on α^2, and therefore may be neglected in comparison with quantities depending upon α, and if the body, supposed at rest, be then left to itself, it will (on the supposition that the equilibrium is stable) oscillate about a horizontal axis through G.

It would in fact come to the same thing if the initial displacement were about G, as the point C would move sensibly (that is, considering small quantities of the first order only,) in a horizontal direction, and the quantity of fluid displaced would, as before, remain unchanged.

If M be the metacentre, the moment of the fluid pressure about G.

$$= g\rho V \,.\, MG \,.\, \sin\theta,$$

and tends to diminish θ, the angle made by GH with the vertical at the time t.

But $$MG = \frac{k^2 A}{V} - a\text{; if } HG = a,$$

therefore, since the horizontal axis through G is a principal axis, we have

$$mK^2 \frac{d^2\theta}{dt^2} = -g\rho\,(k^2 A - aV)\,\theta,$$

neglecting higher powers of θ, where mK^2 is the moment of inertia of the body about the horizontal axis through G,

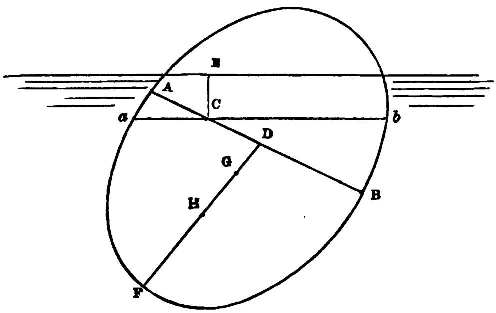
$$\text{or } K^2 \frac{d^2\theta}{dt^2} + g\left(\frac{k^2 A}{V} - a\right)\theta = 0,$$

an equation which, when $k^2 A > aV$, that is, when M is above G, indicates small oscillations taking place in the time

$$\pi K \sqrt{\left\{\frac{V}{g\,(k^2 A - aV)}\right\}}.$$

If G is below H the sign of a will of course be changed.

91. Secondly, when the line joining H and G does not pass through C, the two motions are not independent, but the law which defines these motions can be determined as follows.



Suppose the body to be slightly displaced in the vertical plane of symmetry, and then left to itself; and at the time t let θ be the angle made by HG with the vertical, and $z = CE$ the depth of C below the surface.

Let HG meet the plane of floatation in D,

$$HG = a, \quad CD = b, \quad DG = c,$$

and other symbols as before.

Then the depth of $G = z + b \sin\theta + c\cos\theta$

$$= z + b\theta + c, \text{ to the order considered.}$$

The weight of the fluid displaced is the weight of a volume of fluid equal to

$$aFb + EC, \text{ or } AFB + EC;$$

$$\text{this weight} = g\rho V + g\rho Az,$$

$$\text{and } \therefore m\frac{d^2}{dt^2}(z + c + b\theta) = mg - (g\rho V + g\rho Az)$$

$$= -g\rho Az;$$

$$\text{or } \frac{d^2z}{dt^2} + b\frac{d^2\theta}{dt^2} = -g\frac{A}{V}.z \qquad \text{(I).}$$

Another equation is to be obtained from the consideration of the angular motion about the horizontal axis through G, which is a principal axis, perpendicular to the plane of displacement.

The moment of the fluid pressure about G may be divided into two parts, the one due to the portion aFb, and the other to the portion EC of the fluid displaced.

The former part of the fluid pressure $= g\rho V$ acting upwards through M the metacentre; and the latter $= g\rho Az$, and may be considered to act through C the centre of gravity of the plane of floatation.

The moment, in the direction tending to diminish θ,

$$= g\rho V . GM \sin\theta - g\rho Az(b\cos\theta - c\sin\theta)$$

$$= g\rho(k^2A - aV)\theta - g\rho Az(b - c\theta)$$

$$= g\rho(k^2A - aV)\theta - g\rho Abz,$$

neglecting the product of z and θ;

$$\therefore mK^2\frac{d^2\theta}{dt^2} = -g\rho(k^2A - aV)\theta + g\rho Abz.$$

$$K^2\frac{d^2\theta}{dt^2} = -g\left(\frac{k^2A}{V} - a\right)\theta + g\frac{A}{V}.bz \qquad \text{(II).}$$

From the equations (I) and (II) we obtain

$$\frac{d^2z}{dt^2} + \frac{gA}{V}\left(1 + \frac{b^2}{K^2}\right)z - \frac{gb}{K^2}\left(\frac{k^2A}{V} - a\right)\theta = 0,$$

$$\frac{d^2\theta}{dt^2} - \frac{gAb}{VK^2}z + \frac{g}{K^2}\left(\frac{k^2A}{V} - a\right)\theta = 0,$$

which may be written

$$\frac{d^2z}{dt^2} + rz - bn\theta = 0,$$

$$\frac{d^2\theta}{dt^2} - \frac{pz}{b} + n\theta = 0.$$

To integrate these equations, multiply the second by λ, and add it to the first, then,

assuming $$\frac{\lambda n - bn}{rb - \lambda p} = \frac{\lambda}{b} \quad \text{(III)},$$

we have $$\frac{d^2}{dt^2}(z + \lambda\theta) + \left(r - \frac{\lambda p}{b}\right)(z + \lambda\theta) = 0,$$

and, if λ_1, λ_2 be the roots of (III),

$$z + \lambda_1\theta = C_1 \cos\left\{\sqrt{r - \lambda_1\frac{p}{b}}\,t + \alpha_1\right\},$$

$$z + \lambda_2\theta = C_2 \cos\left\{\sqrt{r - \lambda_2\frac{p}{b}}\,t + \alpha_2\right\},$$

from which z and θ are completely determined.

The depth of G is given by an expression of the form

$$C + A\cos(\mu t + \alpha) + B\cos(\mu' t + \beta),$$

and its motion consists of two distinct oscillations, each following the pendulum laws, and compounded together in accordance with the principle of the coexistence of small oscillations*.

* Poisson's *Cours de Mécanique*, Art. 618.

It may be observed that if two points be taken in the line AB, whose distances from C in the direction CD are λ_1, λ_2, then at the time t, the vertical depths of these points are $z + \lambda_1\theta$ and $z + \lambda_2\theta$, that is, are

$$C_1 \cos\left\{\sqrt{r - \lambda_1 \frac{p}{b}}\, t + \alpha_1\right\}, \text{ and } C_2 \cos\left\{\sqrt{r - \lambda_2 \frac{p}{b}}\, t + \alpha_2\right\},$$

and their vertical motions are therefore simple oscillations following the pendulum law. This remark is quoted by Duhamel (*Cours de Mécanique*, Art. 152) as due to M. Cauchy.

EXAMPLES.

1. A straight rod is dropped vertically from a given height above the surface of water; determine its motion and find the condition that it may be only just immersed.

2. A vertical cylinder floats in a liquid of twice its own density contained in a cylindrical vessel. If the radius of the vessel be double that of the cylinder, and the cylinder be slightly displaced in a vertical direction, find the time of an oscillation.

3. A solid, the lower portion of whose surface is spherical, floats in a heavy fluid; shew that the time of a small angular oscillation is the same in whatever fluid it floats.

4. A hollow hemisphere moveable about a horizontal diameter is partly filled with fluid; shew that the time of a small oscillation is the same as if there were no fluid in it.

5. A solid ellipsoid floats in a liquid of twice its own specific gravity with its shortest axis vertical; find the time of a small vertical oscillation, and also the times of small angular oscillations about the two horizontal axes.

6. A cube (the length of whose edge is $2a$) is floating in a fluid with its centre of gravity at a depth c below the

surface; if it receive a small displacement so that two of its faces remain vertical, shew that the times of its small vertical and angular oscillations are

$$\pi\sqrt{\left(\frac{a+c}{g}\right)} \text{ and } 2\pi\sqrt{\left\{\frac{a^2(a+c)}{g(3c^2+a^2)}\right\}}, \text{ respectively.}$$

7. A candle of S.G. ρ floats vertically in still water of S.G. σ. It is lighted and the flame is observed to descend towards the water with uniform velocity u, and the velocity with which the candle burns is v: prove that

$$v = \frac{\sigma u}{\sigma - \rho}.$$

Prove also, that if the flame be extinguished when a length l of candle remains, the candle will rise out of the water if v be $> \sqrt{\frac{\sigma l g}{\rho}}$; but if v be $< \sqrt{\frac{\sigma l g}{\rho}}$ the time of an oscillation will be $= 2\pi\sqrt{\frac{\rho l}{\sigma g}}$.

8. A right cone is floating with its axis vertical and vertex downwards in a fluid, and $\frac{1}{n}$th part of the axis is immersed; a weight equal to the weight of the cone is placed on the base, upon which the cone sinks till its axis is totally immersed, before rising, shew that

$$n^3 + n^2 + n = 7.$$

9. A cone of vertical angle 2α floats in a cylinder of radius a with a length h of its axis immersed. If it be pushed vertically downwards through a small space, shew that the time of an oscillation is

$$\pi\sqrt{\frac{(a^2 - h^2\tan^2\alpha)\,h}{3a^2 g}}.$$

10. A solid cone, of given vertical angle, is supported on an axis, about which it is moveable, coincident with a

diameter of its base; if the axis be held horizontally, and lowered until one-eighth of the volume of the cone, vertex downwards, is immersed in homogeneous liquid, find the ratio of the densities of the liquid and cone, when the equilibrium is neutral.

If, in the previous case, the axis be not lowered so far as to make the equilibrium neutral, and the cone be then slightly displaced, find the time of a small oscillation.

11. An oblate spheroid is completely immersed in two fluids, the specific gravity of the lower being twice that of the upper fluid, and floats with its axis vertical, and its centre in the common surface of the fluids.

Supposing a small displacement to take place, 1st, in a vertical direction, 2ndly, about a horizontal line through its centre of gravity, shew that the times of the small oscillations will be respectively

$$\pi\sqrt{\left(\frac{2b}{g}\right)}, \text{ and } \pi\sqrt{\left(\tfrac{8}{5}\cdot\frac{b}{g}\,\frac{a^2+b^2}{a^2-b^2}\right)},$$

where a and b are the semi-axes of the generating ellipse.

12. A homogeneous solid floats completely immersed in a liquid, the density of which varies as the depth, with its centre of gravity at a depth h; prove that the time of a small vertical oscillation is

$$2\pi\sqrt{\frac{h}{g}}.$$

13. A lamina of uniform thickness, in the form of an isosceles right-angled triangle, has one of the acute angles fixed below the surface of a fluid, and rests with the side which is not immersed horizontal. Prove that the time of a small oscillation in its own plane is

$$2\pi\sqrt{\left(\frac{a}{g}\right)},$$

where a is the length of each of the sides of the triangle.

14. A solid generated by the revolution of the curve, $y \propto x^{\frac{n}{2}-1}$, about the axis of x, floats with a portion h of the axis immersed; if the solid be depressed through $(n^{\frac{1}{n-1}}-1)h$, it will, on its return, just emerge.

15. A solid of revolution of mass m floats in different liquids. If the time of vertical oscillation in any liquid and its density ρ are found to be connected by the equation

$$\frac{1}{\rho}=f\left(\frac{1}{t^2\rho}\right),$$

f denoting a given function, shew that the equation to the meridian section of the solid is

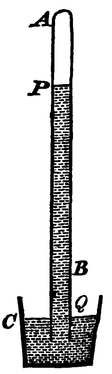
$$x+c=\frac{g}{2\pi m^2}\int\frac{dy}{y}f'\left(\frac{gy^2}{4\pi m}\right).$$

CHAPTER VII.

PRESSURE OF THE ATMOSPHERE.

92. If a glass tube, about three feet in length, having one end closed, be filled with mercury, and then inverted in a vessel of mercury so as to immerse its open end, it will be found that the mercury will descend in the tube, and rest with its upper surface at a height of about 29 inches above the surface of the mercury in the vessel: this experiment, first made by Torricelli, has suggested the use of the *Barometer*, for the purpose of measuring the atmospheric pressure.

The *Barometer*, in its simplest form, is a straight glass tube AB, containing mercury, and having its lower end immersed in a small cistern of mercury; the end A is hermetically sealed, and there is no air in the branch AB.



It is found that the height of the surface P of the mercury above the surface C is about 29 inches, and, as there is no pressure on the surface P, it is clear that the pressure of the air on C is the force which sustains the column of mercury PQ.

We have shewn that the pressure of a fluid at rest is the same at all points of the same horizontal plane; hence the pressure at C is equal to the pressure of the mercury at Q.

Let σ be the density of mercury, and Π the atmospheric pressure at C, then

$$\Pi = g\sigma PQ,$$

and the height PQ measures the atmospheric pressure.

On account of its great density, mercury is the most convenient fluid which can be employed in the construction of barometers, but the pressure of the air may be measured by using any kind of liquid. The density of mercury is about 13·568 times that of water, and therefore the height of the column of water in the water-barometer would be about $33\frac{3}{4}$ feet.

The density of mercury changes with the temperature, and σ must therefore be expressed as a function of the temperature.

Experiment shews that, for an increase of 1° centigrade, the expansion of mercury is $\frac{1}{5550}$th of its volume; hence if σ_t be the density at a temperature t°, and σ_0 at a temperature 0°,

$$\sigma_0 = \sigma_t\left(1 + \frac{t}{5550}\right) = \sigma_t(1 + \cdot00018018t);$$

$$\therefore\ \sigma_t = \sigma_0(1 - \theta t) \text{ if } \theta = \cdot00018018,$$

and $$\Pi = g\sigma_0(1 - \theta t)PQ.$$

By means of the formula, $\Pi = g\sigma_0(1 - \theta t)h$, the atmospheric pressure at any place can be calculated, making due allowance for the change in the value of g consequent on a change of latitude. It is found that this pressure is variable at the same place, with or without changes of temperature, and that in ascending mountains, or in any way rising above the level of the place, the pressure diminishes. This is in accordance with the theory of the equilibrium of fluids, for, in ascending, the height of the column of air above the barometer is diminished, and the pressure of the air upon C, which is equal to the weight of the superincumbent column of air, is therefore diminished, and the mercury must descend in the tube.

If then a relation be found between the height of the mercury and the height through which an ascent has been made, it is clear that by observations, at the *same* time, of the barometric columns at two stations, we shall be able to determine the difference of their altitudes.

We shall investigate a formula for this purpose; but it is first necessary to state the laws which regulate the pressures of the air and gases at different temperatures, and also the laws of the mixture of gases.

93. We have before stated the relation

$$p = k\rho\,(1 + \alpha t)$$

between the pressure, density, and temperature of an elastic fluid: it is deduced from the two following results of experiment:

(1) *If the temperature be constant, the pressure of air varies inversely as its volume.* (*Boyle's Law.*)

(2) *If the pressure remain constant, an increase of temperature of* 1° *C. produces in a mass of air an expansion* ·003665 *of its volume at* 0° *C.* (*Dalton's and Gay-Lussac's Law.*)

Hence, if p be the pressure and ρ_0 the density of air, at a temperature zero,

$$p = k\rho_0.$$

Suppose now the temperature increased to t, the pressure remaining the same: the conception of this may be assisted by considering the air to be contained in a cylinder in which a moveable piston fits closely, and has applied to it a constant force, so that an increase of the elastic force of the air would have the effect of pushing out the piston, until the equilibrium is restored by the diminution of density, and consequent diminution of pressure: we shall then have from the 2nd law,

$$\rho_0 = \rho\,(1 + \alpha t),$$

taking ρ as the new density and $\alpha = \cdot 003665$;

$$\therefore\ p = k\rho\,(1 + \alpha t).$$

If p', ρ' be the pressure and density of the same fluid at a temperature t',

$$p' = k\rho'\,(1 + \alpha t'),$$

and

$$\frac{p}{p'} = \frac{\rho}{\rho'}\,\frac{1 + \alpha t}{1 + \alpha t'}.$$

The quantity α is very nearly the same for gases of all kinds, but k has different values for different gases, and must of course be determined experimentally in every case*.

Absolute Temperature.

94. If we imagine the temperature of a gas lowered until its pressure vanishes, without any change of volume, we arrive at what is called the absolute zero of temperature and absolute temperature is measured from this point.

Assuming t_0 to represent this temperature on the centigrade thermometer, we obtain, from the equation $1 + \alpha t_0 = 0$,

$$t_0 = -\frac{1}{\alpha}, = -273°.$$

In Fahrenheit's scale the reading for absolute zero is $-459°$.

The equations,

$$p = \kappa\rho(1 + \alpha t),$$

$$0 = \kappa\rho(1 + \alpha t_0),$$

lead to

$$p = \kappa\rho\alpha(t - t_0),$$

$$= \kappa\rho\alpha T,$$

if T be the absolute temperature.

Since ρV is constant, it follows that $\frac{pV}{T}$ is constant, and this law expresses, in the absolute scale, the relation between pressure, volume, and temperature.

The pressure of a mixture of different elastic fluids.

95. Consider two different gases, contained in vessels of which the volumes are V and V', and let their pressures and temperatures, p and t, be the same.

Let a communication be established between the two vessels, or transfer both the gases to a closed vessel, the volume of which is $V + V'$: it is found that, unless a chemical action take place, the two gases do not remain separate,

* Methods of determining the value of α are described in Deschanel's *Natural Philosophy*, translated and edited by Professor Everett.

but permeate each other until they are completely mixed, and that, when equilibrium is attained, the pressure and temperature are the same as before. From this important experimental fact we can deduce the following proposition.

If two gases having the same temperature be mixed together in a vessel, the volume of which is V, *and if the pressure of the two gases. alone filling the volume* V, *be* p *and* p′, *the pressure of the mixture will be* p + p′.

Suppose the two gases separated; let the gas, of which the pressure is p, have its volume changed, without any alteration of temperature, until its pressure becomes p'; its volume will be, by Marriotte's law, $\frac{p}{p'}V$.

Let the two gases be now mixed in a vessel, of which the solid content is

$$V+\frac{p}{p'}V, \text{ or } \frac{p+p'}{p'}V;$$

the pressure of the mixture will still be p', and the temperature will be unaltered. If the mixture be then compressed into a volume V, its pressure will become, by the application again of Marriotte's law, $p+p'$.

This result is obviously true for a mixture of any number of gases.

96. *Two volumes* V, V′ *of different gases, at pressures* p, p′ *respectively, are mixed together, so that the volume of the mixture is* U; *to find the pressure of the mixture.*

The pressures of the two gases, reduced to the volume U, are respectively

$$\frac{V}{U}p, \ \frac{V'}{U}p',$$

and therefore, by the preceding article, the pressure of the mixture is

$$\frac{V}{U}p+\frac{V'}{U}p';$$

and if ϖ be this pressure, we have

$$\varpi U = pV + p'V'.$$

97. The laws and results of the preceding articles are equally true of vapours, the only difference between the mechanical qualities of vapours and gases, irrespective of their chemical characteristics, being that the former are easily condensed into liquids by lowering the temperature, while the latter can only be condensed by the application either of great pressure or extreme cold, or of a combination of both*.

98. If water be introduced into a space containing dry air, vapour is immediately formed, and it is found that the pressure and density of the vapour are dependent only on the temperature, and are quite independent of the density of the air, and indeed are exactly the same if the air be removed. If the temperature be increased or the space enlarged, an additional quantity of vapour will be formed, but if the temperature be lowered or the space diminished, some portion of the vapour will be condensed.

While a sufficient quantity of water remains, as a source from which vapour is supplied, the space will be always saturated with vapour, that is, there will be as much vapour as the temperature admits of; but if the temperature be so raised that all the water is turned into vapour, then for that, and all higher temperatures, the pressure of the vapour will follow the same law as the pressure of the air.

In any case, whether the space be saturated or not, if p be the pressure of the air, and ϖ of the vapour, the pressure of the mixture is $p + \varpi$.

99. The atmosphere always contains aqueous vapour, the quantity being greater or less at different times; if any portion of the space occupied by the atmosphere be saturated with vapour, that is, if the density of the vapour be as great

* Professor Faraday succeeded in condensing carbonic acid gas, and other gases requiring a considerable pressure for the purpose, and the result of his experiments led to the conclusion that, in all probability, all gases are the vapours of liquids. This conclusion was remarkably supported in 1877, when M. Pictet, in the early part of the year, liquefied oxygen by applying to it a pressure of 300 atmospheres, and, in December of the same year, M. Cailletet liquefied nitrogen, hydrogen, and atmospheric air.

as it can be for the temperature, then any reduction of temperature will produce condensation of some portion of the vapour, but if the density of the vapour be not at its maximum for that temperature, no condensation will take place until the temperature is lowered below the point corresponding to the saturation of the space.

Formation of Dew. If any surface, in contact with the atmosphere, be cooled down below the temperature corresponding to the saturation of the space near it, condensation of the aqueous vapour will ensue, and the condensed vapour will be deposited in the form of *dew* upon the surface. The formation of dew on the ground depends therefore on the cooling of its surface, and this is in general greater and more quickly effected, when the sky is free from clouds, and when, consequently, the loss of heat by radiation is greater than under other circumstances.

The *Dew Point* is the temperature at which dew first begins to be formed, and must be determined by actual observation.

The pressure of vapour corresponding to its saturating densities for different temperatures must also be determined experimentally, and, if this be effected, an observation of the dew point at once determines the pressure of the vapour in the atmosphere. For if t' be the dew point, and p' the known corresponding pressure, then at any other temperature t above t' the pressure p is given by the equation

$$\frac{p}{p'} = \frac{1 + \alpha t}{1 + \alpha t'}.$$

100. *Effect of compression or dilatation on the pressure and temperature of a gas.*

It is found by experiment that if a quantity of air, enclosed in a vessel impervious to heat, be compressed, its temperature is raised; and that, if a quantity of air, enclosed in any kind of vessel, be suddenly compressed, so that there is no time for the heat to escape, the temperature is similarly raised.

101. *Thermal capacity.*

The thermal capacity of a body is measured by the amount of heat required to raise the temperature one degree.

The unit of heat which is actually employed is the quantity of heat required to increase by one degree one unit of mass of water, supposed to be between 0°C. and 40°C.

Specific Heat.

The specific heat of a body is the thermal capacity of one unit of mass, or, which is the same thing, it is the ratio of the amount of heat required to increase by 1° the temperature of the body to the amount of heat required to increase by 1° the temperature of an equal weight of water.

If an amount of heat dQ produce in the unit of mass a change of temperature dt, the measure of the specific heat is $\frac{dQ}{dt}$.

In gases it is necessary to consider two cases; (1) when the pressure remains constant, the gas being allowed to expand, (2) when the volume remains constant.

We shall denote the specific heat in these two cases by the symbols c_p and c_v.

It is easy to see that c_p is greater than c_v, for in the first case the heat imparted does work in expanding the gas as well as in raising its temperature.

102. To determine the effect of a compression or a dilatation of a given quantity of gas, it is clear to begin with that the heat required will be a function of v, p, and T, and since $pv \propto T$, the heat required for any expansion will be a function of v and p. Therefore it follows that

$$dQ = \frac{dQ}{dv}\,dv + \frac{dQ}{dp}\,dp,$$

and, in general, $p = k\rho\alpha T$ or, if the mass of the given quantity of gas be the unit of mass,

$$pv = k\alpha T = KT.$$

If the pressure be constant, $dQ = c_p dT$;

$$\therefore \frac{dQ}{dv} dv = c_p dT = c_p \frac{pdv}{K}$$

and
$$\frac{dQ}{dv} = \frac{c_p}{K} p.$$

If the volume be constant,

$$\frac{dQ}{dp} dp = c_v dT = c_v \frac{vdp}{K}$$

and
$$\frac{dQ}{dp} = \frac{c_v}{K} v.$$

Therefore, if no heat be imparted, that is, if $dQ = 0$,

$$c_p \frac{dv}{v} + c_v . \frac{dp}{p} = 0;$$

$$\therefore p^{c_v} . v^{c_p} \text{ is constant,}$$

or
$$pv^\gamma \text{ is constant, if } \gamma = \frac{c_p}{c_v}.$$

If p, v be changed to p', v', we obtain

$$\frac{p'}{p} = \left(\frac{v}{v'}\right)^\gamma = \left(\frac{c'}{c}\right)^\gamma,$$

and
$$\frac{t'}{t} = \frac{p'v'}{pv} = \left(\frac{v}{v'}\right)^{\gamma-1}.$$

The equation $pv^\gamma =$ constant is, in thermodynamics, the equation of the adiabatic, or isentropic lines, and it represents the relation between the pressure and volume of a mass of air which is suddenly compressed or dilated. It will be found subsequently that this relation is of great importance in the theory of sound.

103. It can be shewn by the aid of the principle of energy, that the difference between c_p and c_v, for any given gas, is constant.

By a law of thermodynamics, the energy imparted to a system by the application of heat is proportional to the amount of heat.

Hence, J being the mechanical equivalent of heat, that is, the work done by the unit of heat, the energy imparted to the unit mass of a gas by a rise of temperature dT when the pressure is constant is

$$J \,.\, c_p dT.$$

But this energy is partly expended in elevating the temperature at a given volume, and partly in expanding the volume;

$$\therefore\; J \,.\, c_p dT = pdv + J \,.\, c_v dT$$

and
$$pv = KT,$$

$$\therefore\; J\,(c_p - c_v) = K,$$

shewing that $c_p - c_v$ is constant.

We can employ this equation in obtaining the result of Art. 102.

For if no heat be supplied, no energy is imparted,

and
$$\therefore\; pdv + J \,.\, c_v dT = 0.$$

But
$$pv = K\alpha T = J \,.\, (c_p - c_v)\, T;$$

$$\therefore\; pdv + vdp = J \,.\, (c_p - c_v)\, dT,$$

and
$$pdv\,(c_p - c_v) + c_v\,(pdv + vdp) = 0,$$

whence
$$c_p \,.\, pdv + c_v \,.\, vdp = 0, \text{ as before.}$$

Whole mass of the Earth's Atmosphere.

104. Some idea may be formed of the mass of air and vapour surrounding the earth by means of the barometer. Supposing the earth to be a sphere of radius r, and that the height of the barometric column, h, is the same at all points of its surface, the mass of the atmosphere is approximately equivalent to the mass $4\pi\sigma r^2 h$ of mercury.

Let ρ be the mean density of the earth;

then, the mass of the atmosphere : the mass of the earth

$$= 4\pi\sigma r^2 h : \rho \tfrac{4}{3}\pi r^3$$
$$= 3\sigma h : \rho r.$$

But, taking water as the standard substance, $\sigma = 13{\cdot}57$, and ρ has been found to be about $5{\cdot}5$; and, if we take $29{\cdot}9$ inches as an approximate value of h, it will be found that the ratio of the masses is somewhat less than the ratio of one to a million*.

The height of the homogeneous atmosphere.

105. If the whole column of air had the same density throughout as at the surface, its height being l, and the height of the mercury being h, we should have

$$\sigma h = \rho l,$$

where ρ is the density of the air. It has been found that the ratio $\sigma : \rho$ is about $10462 : 1$, and therefore, employing as before $29{\cdot}9$ as a value of h, it will be found that l is a little less than 5 miles.

Necessary limit to the height of the atmosphere.

It is clear that, since at a distance from the earth's surface its attraction diminishes, and the density and pressure of the air are therefore diminished, the above result is very far from the truth. A *limit* to the height can however be found from the consideration that, beyond a certain distance from the earth's centre, its attraction will be unable to retain the particles of air in the circular paths, which they must describe about the earth, in order to remain in a state of relative equilibrium.

* The observations on the motion of pendulums, made by Sir G. B. Airy at the Harton Colliery in 1854, have thrown doubt on the accuracy of the value 5·5, which has been assumed, in Art. 94, as a measure of the mean density of the earth.

The value deduced from the Harton *Observations* is 6·566 with a probable error ± ·0182. *Phil. Trans.* 1856.

At the equator the expression $\omega^2 r$, ω being the earth's angular velocity, is equal to $\frac{g}{289}$, and therefore, at a height z, the force necessary to retain a particle m of air in its circular motion is equal to $\frac{mg}{289}\frac{r+z}{r}$; the earth's attraction at the same height

$$=\frac{mgr^2}{(r+z)^2};$$

and the extreme height is given by the equation

$$\frac{r^2}{(r+z)^2}=\frac{r+z}{289\,r}$$

or $$z=r\{\sqrt[3]{(289)}-1\},$$

that is, z is a little greater than $5r$.

It is possible however that this height is considerably beyond the true height, for the temperature of the air has been found, by experiments made in balloons, to diminish with great rapidity during an ascent, and it is therefore quite possible, that, at a height less than $5r$, the air may be liquefied by extreme cold, and its external surface would be in that case, of the same kind as the surfaces of known inelastic fluids.

The determination of heights by the barometer.

106. Consider a vertical column of the atmosphere at rest under the action of gravity: at a height z let p be the pressure and ρ the density, and at a height $z+\delta z$, let $p+\delta p$ be the pressure.

If A be the area of the section of the column, the volume $A\delta z$ of air may be considered as in equilibrium under the action of the pressures pA and $(p+\delta p)A$, and of its weight $g\rho A\delta z$.

Hence we have $$\delta p=-g\rho\delta z;$$

and, if t be the temperature, $p=k\rho(1+\alpha t)$;

$$\therefore \text{ in the limit } \frac{k}{p} \cdot \frac{dp}{dz} = -\frac{g}{1+\alpha t}.$$

We shall suppose t constant, and therefore

$$k \log p = -\frac{gz}{1+\alpha t} + C,$$

and, if p' be the pressure at a height z', we obtain

$$k \log \frac{p}{p'} = \frac{g(z'-z)}{1+\alpha t}.$$

Let h, h', be the observed heights of the barometer at two stations, the heights of which are z and z'; then, taking σ as the density of mercury at a temperature zero, and τ, τ', as the temperatures at the two stations,

$$p = g\sigma h(1-\theta\tau), \text{ and } p' = g\sigma h'(1-\theta\tau');$$

$$\therefore z'-z = \frac{k}{g}(1+\alpha t) \log \frac{h(1-\theta\tau)}{h'(1-\theta\tau')};$$

t may be taken as approximately equal to $\frac{1}{2}(\tau+\tau')$, and we thus have an equation from which the difference of the heights of the two stations can be calculated.

107. If however the heights above the earth's surface be considerable, it is necessary to take account of the variation of gravity at different distances from the earth's centre. We proceed then to an investigation of a more exact formula.

Let g be the measure of gravity at the level of the sea, and r the radius of the earth, then, at a height z, the attractive force is measured by

$$g\frac{r^2}{(r+z)^2},$$

and the equation of equilibrium is

$$dp = -g\frac{r^2}{(r+z)^2}\rho dz;$$

we have also $p = k\rho(1+\alpha t)$, and it is here important to observe that p is the sum of the pressures due to the air itself,

and to the aqueous vapour which is mixed with it, so that, if ρ' be the density of the aqueous vapour, p is the sum of two quantities in the form

$$k\rho\,(1+\alpha t)+k'\rho'\,(1+\alpha t),$$

and therefore the quantity $k\rho$ in the above equation is the sum of the two $k\rho$, $k'\rho'$, corresponding respectively to the air and the aqueous vapour.

From the two equations above we obtain

$$k\frac{dp}{p}=-\frac{1}{1+\alpha t}\frac{gr^2dz}{(r+z)^2},$$

and, as before, we shall consider t constant, and equal to the mean of the temperatures at the two stations.

By integration

$$k\log p=\frac{1}{1+\alpha t}\frac{gr^2}{r+z}+C,$$

$$\text{and } \therefore k\log\frac{p'}{p}=\frac{gr^2(z-z')}{(1+\alpha t)(r+z)(r+z')}\;\ldots\ldots\ldots(1).$$

Let h, h', be the observed heights of the mercury, and τ, τ', the temperatures, as before; then, since the force of gravity at a height z is measured by the quantity $\dfrac{gr^2}{(r+z)^2}$, we have

$$p=\frac{gr^2}{(r+z)^2}\sigma h\,(1-\theta\tau),$$

$$p'=\frac{gr^2}{(r+z')^2}\sigma h'\,(1-\theta\tau'),$$

$$\frac{p'}{p}=\left(\frac{r+z}{r+z'}\right)^2\frac{1-\theta\tau'}{1-\theta\tau}\frac{h'}{h}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and therefore, observing that θ is a very small quantity,

$$z-z'=$$

$$\frac{k\,(1+\alpha t)(r+z)(r+z')}{\mu gr^2}\left\{\log_{10}\frac{h'}{h}+2\log_{10}\frac{r+z}{r+z'}-\mu\theta\,(\tau'-\tau)\right\},$$

where $\mu=\log_{10}\epsilon=\cdot4342945$.

From this formula, if z' be known, the value of z can be calculated.

If the lower station be nearly at the level of the sea, $z' = 0$, and

$$z = \frac{k\,(1+\alpha t)}{\mu g}\left(1+\frac{z}{r}\right)\left\{\log_{10}\frac{h'}{h} + 2\log_{10}\left(1+\frac{z}{r}\right) - \mu\theta\,(\tau'-\tau)\right\}\ldots(3).$$

108. In the preceding investigation we have taken no account of the variation of gravity at different parts of the earth's surface; but if g' be the measure of gravity at a place of which the latitude is λ', and g at a place of latitude λ, it has been found, (Poisson, Art. 628), that

$$\frac{g}{g'} = \frac{1 - \cdot 002588\cos 2\lambda}{1 - \cdot 002588\cos 2\lambda'}:$$

the value of g obtained from this equation, in which g' and λ' are supposed to be known, must be employed in the above formula.

If λ' be the latitude of Paris, the value of the quantity

$$\frac{k}{\mu g'}\,(1 - \cdot 002588\cos 2\lambda')\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

is nearly 18336 French metres or about 60158·56 English feet*, and, representing this numerical quantity by c, the expression for z becomes

$$\frac{c\,(1+\alpha t)\left(1+\frac{z}{r}\right)}{1-\cdot 002588\cos 2\lambda}\left\{\log_{10}\frac{h'}{h} + 2\log_{10}\left(1+\frac{z}{r}\right) - \mu\theta\,(\tau'-\tau)\right\}\ldots(5).$$

The value of c may be obtained by direct calculation of the expression (4), and the calculated value is 18337·46 metres; it has been found however, by comparing the results of trigonometrical measurements with the results of the formula (5), that 18336 metres is a more accurate value of the coefficient.

* A French metre is 39·37079 inches.

In order to calculate z from the formula (5), an approximate value must be first obtained by neglecting $\frac{z}{r}$ in the right-hand member of the equation; if this approximate value be then employed in the same expression, a more accurate value will result, and the same process may, if necessary, be repeated.

109. Other corrections are however necessary in order to render the determination of heights by the barometer very exact in practice; the value of k for instance is modified by the fact that the density of aqueous vapour at a given temperature and pressure is less than the density of dry air under the same circumstances, and the proportion of aqueous vapour to dry air may be, and in general will be, different at the two stations.

Moreover, if the upper station be on the surface of the ground, the attraction of the portion of the earth which is above its mean level must be taken account of. The effect of this attraction is to increase the quantity $\frac{gr^2}{(r+z)^2}$ by $\frac{3gz}{4r}$, so that, at a height z, the force of gravity is measured by

$$\frac{gr^2}{(r+z)^2}+\frac{3gz}{4r},$$

or, approximately, $g\left\{1-\frac{5z}{4r}\right\}$, (Poisson, *Mécanique*, Art. 629); the equation for p will be in this case

$$dp=-g\left\{1-\frac{5z}{4r}\right\}\rho dz,$$

and therefore, if the lower station be at the level of the sea,

$$k(1+\alpha t)\log\frac{p'}{p}=gz\left(1-\tfrac{5}{8}\frac{z}{r}\right)$$

or
$$z=\frac{k(1+\alpha t)}{g}\left(1+\tfrac{5}{8}\frac{z}{r}\right)\log\frac{p'}{p}.$$

In place of the equation (2) we shall have

$$\frac{p'}{p}=\left(1+\tfrac{5}{4}\frac{z}{r}\right)\frac{1-\theta\tau'}{1-\theta\tau}\frac{h'}{h},$$

and the final equation for z will be obtained by substituting in (5),

$$1 + \tfrac{5}{8}\frac{z}{r} \text{ for } 1 + \frac{z}{r}, \text{ observing that } \log\left(1 + \tfrac{5}{4}\frac{z}{r}\right)$$

$$\text{is approximately equal to } 2\log\left(1 + \tfrac{5}{8}\frac{z}{r}\right).$$

When $\frac{z}{r}$ is very small, it may be neglected in the formula (5). It has however been found in practice that the results are rendered more accurate, for such cases, by employing, as the value of c, 18393 metres. (Duhamel, p. 259.)

110. In the preceding articles we have supposed the temperature of the air to be constant through the whole of the vertical space between the two stations; if however the difference between the heights be very great, a considerable error may be thus introduced, and formulæ have therefore been constructed in which account is taken, on various hypotheses, of the variation of atmospheric temperature. A formula of this kind is given in Lindenau's Barometric Tables, constructed on the supposition that the temperature diminishes in harmonic progression through a series of heights increasing in arithmetic progression.

It must also be noticed that we have assumed the temperature of the mercury in the barometer to be the same as that of the air surrounding it; but in some cases, as for instance when observations are made in a balloon, the barometer may not remain long enough in the same place to acquire the temperature of the air round it. The temperature of the mercury can, however, be observed by a thermometer the bulb of which is placed in the cistern of the barometer, and the temperatures so obtained must be employed in the equation (2) of Art. (107).

111. The two following problems are illustrative of the principles of this chapter.

(1) *A piston without weight fits into a vertical cylinder, closed at its base and filled with atmospheric air, and is initially at the top of the cylinder; water being poured slowly*

on the top of the piston, find how much can be poured in before it will run over.

Let a be the height of the cylinder, and z the depth to which the piston will sink; then in the position of equilibrium the pressure of the air in the cylinder is $\Pi + g\rho z$, where Π is the atmospheric pressure, and ρ the density of water: but

$$\text{this pressure} : \Pi :: a : a - z;$$

$$\therefore \frac{\Pi a}{a - z} = \Pi + g\rho z.$$

Let h be the height of the water-barometer,

$$\therefore \Pi = g\rho h,$$

$$ha = (a - z)(h + z),$$

and $$z = 0 \text{ or } a - h.$$

Unless then the height of the cylinder is greater than h, no water can be poured in, for, even if the piston be forced down and water then poured on it, the pressure of the air beneath will raise the piston.

The negative solution, when $a < h$, can however be explained as the solution of a different problem leading to the same algebraic equation. Suppose the cylinder to be continued above the piston, and let it be required to raise the piston through a space z by a force which shall be equal to the weight of the cylindrical space z of water.

This leads to the equation

$$\frac{\Pi - g\rho z}{\Pi} = \frac{a}{a + z},$$

$$\text{or } z = h - a.$$

(2) *To determine the motion of a balloon on the supposition that the mass of air displaced by it in any position is homogeneous, and that the temperature throughout is constant.*

Let z be the height of the centre of gravity of the balloon, m its mass, V its volume, and ρ the density of the air at the

height z; then the equation which determines the motion is

$$m\frac{d^2z}{dt^2} = g'\rho V - mg',$$

where
$$g' = g\frac{r^2}{(r+z)^2}.$$

But from the equations $dp = -g'\rho dz$ and $p = k\rho$, we obtain

$$p = \Pi\epsilon^{-\frac{grz}{k(r+z)}},$$

and therefore

$$m\frac{d^2z}{dt^2} = \frac{\Pi Vgr^2}{k(r+z)^2}\epsilon^{-\frac{grz}{k(r+z)}} - mg\frac{r^2}{(r+z)^2};$$

from which, putting $m = \sigma V$, multiplying by $2\frac{dz}{dt}$, and integrating,

$$\sigma\left(\frac{dz}{dt}\right)^2 = C - 2\Pi\epsilon^{\frac{-grz}{k(r+z)}} + \frac{2\sigma gr^2}{r+z};$$

initially
$$0 = C - 2\Pi + 2\sigma gr,$$

$$\therefore\ \sigma\left(\frac{dz}{dt}\right)^2 = 2\Pi\{1 - \epsilon^{\frac{-grz}{k(r+z)}}\} - \frac{2\sigma grz}{r+z}.$$

The greatest height of the balloon is given by putting

$$\frac{dz}{dt} = 0,$$

and, if the mean density of the balloon differ very little from that of the air, $\frac{z}{r}$ will be small, and an approximate value may be found.

EXAMPLES.

1. If the density of air be ·0013, that of mercury 13·59, and if the height of the barometer be 30 inches, prove that the numerical value of κ is about 836300, a foot and a second being units of space and time.

2. The weight of 1 litre of dry air at 15·5° C. when the height of the barometer is 760 mm. is 1·23 grammes. The pressure of aqueous vapour at this temperature is 12·6 mm. of mercury, and its density is to that of dry air at the same temperature and pressure as 5 to 8. Find the weight of a litre of air when saturated with aqueous vapour at the above temperature and pressure.

3. A faulty barometer indicated 29·2 and 30 inches when the indications of a correct instrument were 29·4 and 30·3 inches respectively; find the length of tube which the air in the tube would fill under the pressure of 30 inches.

4. The barometer standing at 30 inches, a cubic yard of atmospheric air is compressed into a vessel containing a cubic foot; find approximately the numerical measure of the energy stored up, the specific gravity of mercury being 13·596 referred to water, of which a cubic inch weighs 252·77 grains.

5. The readings of a perfect mercurial barometer are α and β, while the corresponding readings of a faulty one, in which there is some air, are a and b; prove that the correction to be applied to any reading c of the faulty barometer is

$$\frac{(\alpha - a)(\beta - b)(a - b)}{(a - c)(\alpha - a) - (b - c)(\beta - b)}.$$

6. If a thermometer, plunged incompletely in a liquid whose temperature is required, indicate a temperature t, and τ be that of the air, the column not immersed being m degrees, prove that the correction to be applied is $\dfrac{m(t-\tau)}{6840 + \tau - m}$, $\dfrac{1}{6840}$ being the expansion of mercury in glass for 1° of

temperature, assuming that the temperature of the mercury in each part is that of the medium which surrounds it.

7. A closed vertical cylinder of unit sectional area contains a piston, weight W. The piston is originally halfway up the cylinder, and the space above and below is filled with saturated air. On being left to itself the piston sinks to half its former height; prove that the tension of the saturated vapour is $3W-4\Pi$ where Π is the pressure of the atmosphere: the temperature being supposed the same at the end and beginning of the process.

8. A vertical barometer tube is constructed, of which the upper portion is closed at the top, and has a sectional area a^2, the middle portion is a bulb of volume b^3, and the lower portion has a section c^2, and is open at the bottom; the mercury fills the bulb and part of the upper and lower portions of the tube, and is prevented from running out below by means of a float against which the air presses; the upper part of the tube is a vacuum: find the change of position of the upper and lower ends of the mercurial column, due to a given alteration of the pressure of the atmosphere.

Shew also that, if the whole volume of the mercury in the instrument be c^2H, where H is the height of the barometer, the upper surface will be unaffected by changes of temperature.

9. A cylindrical diving-bell sinks in water until a certain portion V remains occupied by air, and in this position a quantity of air, whose volume under the atmospheric pressure was $2V$, is forced into it. Shew how far the bell must sink in order that the air may occupy the same space as in the first position.

Find also the condition that when the air is forced in at the first position no air may escape from beneath the bell.

10. Two equal closed cylinders both contain known quantities of water and air. One is placed above the other, and a communication made between the water in each. Find

the amount which will flow from the upper to the lower before there is equilibrium.

Suppose the whole now introduced into a warm room, which way will the water flow?

11. A hollow cylinder containing air is fitted with an air-tight piston which when the cylinder is placed vertically is at a given height above the base; the cylinder being now inverted and placed vertically in a fluid sinks partly below the surface; find the position of equilibrium.

12. A vessel, in the form of the surface generated by the revolution about its axis of an arc of a parobola terminated by the vertex, is immersed, mouth downwards, in a trough of mercury; shew that the pressure of the air contained in the vessel varies inversely as the square of the distance of the vertex of the vessel from the surface of the mercury within it. Supposing the length of the axis of the vessel to be to the height of the barometer as 45 is to 64, find the depth of the surface of the mercury within the vessel, when the whole vessel is just immersed.

13. A piston without weight fits into a vertical cylinder, closed at its base and filled with air, and is initially at the top of the cylinder; if water be slowly poured on the top of the piston, shew that the upper surface of the water will be lowest when the depth of the water is $\sqrt{(ah)} - h$, where h is the height of the water-barometer, and a the height of the cylinder.

14. The barometer stands at 29·88 inches, and the thermometer is at the Dew Point: a barometer and a cup of water are placed under a receiver, from which the air is removed, and the barometer then stands at ·36 of an inch; find the space which would be occupied by a given volume of the atmosphere, if it were deprived of its vapour without changing its pressure or temperature.

15. A straight tube, closed at one end and open at the other, revolves with a constant angular velocity about an axis meeting the tube at right angles; neglecting the action of gravity, find the density of the air within the tube at any point.

Both ends are open.

16. A bent tube of uniform bore, the arms of which are at right angles, revolves with constant angular velocity ω about the axis of one of its arms, which is vertical and has its extremity immersed in water. Prove that the height to which the water will rise in the vertical arm is

gravity

$$\frac{\Pi}{g\rho}\left(1 - e^{-\frac{\omega^2 a^2}{2k}}\right),$$

a being the length of the horizontal arm, Π the atmospheric pressure, and ρ the density of water.

17. Prove that for rough purposes the difference of the logarithms of the heights of the barometer multiplied by 10000 gives the difference of the heights of two stations in fathoms.

18. Two non-conducting vessels, of volumes v and v', contain atmospheric air at pressures p and p', at the temperatures T and T'; if these masses of air be mixed together in a non-conducting vessel of volume V, find the pressure of the mixture.

19. Two bulbs containing air are connected by a horizontal glass tube of uniform bore, and a bubble of liquid in this tube separates the air into two equal quantities. The bubble is then displaced by heating the bulbs to temperatures t degrees and t' degrees: prove that, if the temperature of each bulb be decreased τ degrees, the bubble will receive an additional displacement which bears to the original displacement the ratio of $2\alpha\tau : 2 + \alpha(t + t' - 2\tau)$, where α is the coefficient of expansion.

20. An elastic spherical envelope is surrounded by air saturated with vapour; when the air within it is at a pressure of two atmospheres it is found that its radius is twice its natural length, and again the radius is three times its natural length when the envelope contains 77 times as much air as it would if open to the air; assuming that the tension at any point varies as the extension of the surface, prove that $\frac{1}{25}$ of the pressure of the air is due to the vapour it contains.

21. A conical shell, vertical angle $\frac{\pi}{2}$, and height H, can

hold double its own weight of water. It is inverted and immersed, axis vertical, in a mass of water. The water is now made to rotate with angular velocity $\sqrt[6]{\frac{7g^3}{2H^3}}$, and the cone sinks till its vertex lies in the surface: prove that the height of the water-barometer is to that of the cone as $3 : \sqrt[3]{28}$.

22. A small balloon containing air is immersed in water and has 100 grains of lead attached to it, the envelope of the balloon being of the same density as the water. If at the temperature of the water and the pressure of the atmosphere the balloon contain 1 cub. inch of air, find the depth to which it must be immersed in the water in order to be in a position of unstable equilibrium when the height of the water barometer is 33 feet; it being given that the density of air : that of water : that of lead :: 1 : 800 : 9120.

23. A cup is formed out of a uniform solid paraboloid, by removing half the volume, so that the inner boundary is an equal coaxal paraboloid with its vertex at the focus of the former one. The cup is immersed in vacuo in a fluid, vertex upwards and axis vertical, and gas is forced in from below till the vertex rises to the surface: if the water be now halfway up the inner boundary of the cup, prove that the density of the fluid is $\frac{4}{3}$ that of the paraboloid.

24. If the pressure of the air varied as the $\left(1+\frac{1}{m}\right)$th power of the density, shew that, neglecting variations of temperature and gravity, the height of the atmosphere would be equal to $(m+1)$ times the height of the homogeneous atmosphere.

25. A piston of weight w rests in a vertical cylinder of transverse section k, being supported by a depth a of air. The piston rod receives a vertical blow P, which forces the piston down through a distance h: prove that

$$(w+\Pi k)\left\{h+a\log\left(1-\frac{h}{a}\right)\right\}+\frac{gP^2}{2w}=0,$$

Π being the atmospheric pressure.

CHAPTER VIII.

THE TENSION OF FLEXIBLE SURFACES.

112. THE general problem of the equilibrium of flexible surfaces is considered by Lagrange, *Mécanique Analytique*, Tom. I., and also, more fully by Poisson, *Mémoires de l'Institut*, 1812; it is proposed in this Chapter to discuss one class of the questions which arise out of the general case, those namely which have reference to the action of fluids upon flexible surfaces.

The pressure of a fluid at rest being normal to any surface with which it is in contact, we have, in fact, to consider the equilibrium of flexible surfaces at rest under the action of normal pressures, and of the tensions at their bounding lines.

For the sake of generality the term 'flexible surface' is employed as the representative of substances, such as cloth and thin paper, which do not offer any sensible resistance to bending, and which, when bent or twisted, do not tend to return to their original form. Perfectly flexible surfaces, whether extensible or inextensible, are therefore to be looked upon as inelastic.

In the following articles we shall suppose that the stress between any two portions of a flexible surface is wholly tangential to the surface.

Measure of Tension.

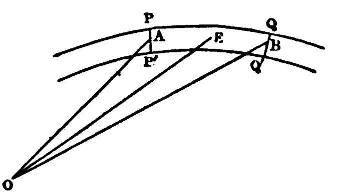
Conceive a flexible and inelastic surface, extensible or inextensible, in a state of tension, and let QPQ' be a small

arc of the section through P made by a normal plane; then if $t \,.\, QQ'$ be the resultant action, perpendicular to QQ' in the tangent plane, between the portions of surface bounded by the line QQ', t is the measure of the tension at P; in other words, t is the rate of tension at P, or the force which would be exerted on a section of the substance, the length of which is unity, in the same state of tension throughout as the surface at P.

In general the stress between the portions of surface separated by QQ' will not be perpendicular to QQ', and will therefore be the resultant of the tension $t \,.\, QQ'$ and of a force $\tau \,.\, QQ'$ tangential to the curve QQ', τ being a quantity of the same kind as t and measured in the same way.

113. *A vessel in the form of a right circular cylinder, the curved surface of which is flexible, contains fluid; the axis of the cylinder being vertical, it is required to find the relation between the pressure and tension at any point.*

Let PQ' be a small portion of the surface contained between two planes perpendicular to the axis and two generating lines of the cylinder.



Let t be the horizontal tension and p the pressure, at any point of PQ, and suppose the element PQ' of the surface to be made rigid; then its equilibrium will be maintained by the normal pressure of the fluid, $pPP' \,.\, PQ$, the tangential forces tPP' and tQQ', and by the vertical tensions on PQ and $P'Q'$, if there be any tension in the vertical direction.

Hence, resolving the forces in the direction of the normal OE, drawn to the middle point E,

$$p \, . \, PP' \, . \, PQ = 2tPP' \sin\left(\tfrac{1}{2} POQ\right),$$

$$= 2tPP' \tfrac{1}{2} \frac{PQ}{r}, \text{ if } r \text{ be the radius,}$$

$$\text{or } t = pr.$$

114. *If fluid at rest under the action of given forces be contained in a cylindrical surface of any form, the tension at any point of a section perpendicular to the axis of the cylinder is the same.*

Let PQ', (figure, Art 113), be an element of the surface, O the centre of curvature at A, t the tension at A, $t + \delta t$ at B, and $\delta\phi$ the angle between the tangents at A and B.

Also, let $\delta\psi$ be the inclination to OA of the direction of the fluid pressure on PQ', which must lie between OA and OB.

Then, resolving along the tangent at A,

$$(t + \delta t)\cos\delta\phi - t = pAB \sin\delta\psi,$$

$$= pr\delta\phi \sin\delta\psi,$$

if r be the radius of curvature at A.

Hence, ultimately, when $\delta\phi$ vanishes,

$$\frac{dt}{d\phi} = 0,$$

and, as this is the case at every point of the section, it follows that t is constant.

By resolving the forces in the direction OA, we shall obtain, as in the previous article, the relation

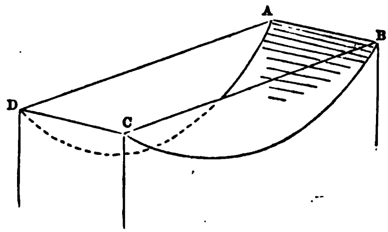
$$t = pr,$$

between the tension perpendicular to the generating line, the pressure, and the curvature, at any point of the surface.

The Lintearia and the Elastica.

115. The Lintearia is the curve formed by pouring water upon a rectangular piece of thin cloth, the ends of which are

supported horizontally, while the water is prevented from escaping at the sides.

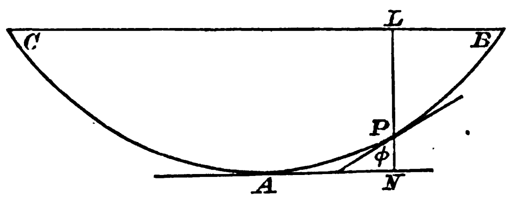


Thus, if the ends AB, CD, of the cloth or membrane be fastened to the sides of a box, and if the sides AD, BC fit the box closely and water be poured in, the cross section of the cloth by a vertical plane parallel to AD or BC is the Lintearia.

The pressure being normal, the tension of the cloth is constant, and therefore, if r be the radius of curvature at P, and BC the surface of the water,

$$g\rho PL \,.\, r \text{ is constant},$$

$$\therefore \frac{c^2}{r} = PL = h - y, \text{ taking } PN = y.$$



Hence,
$$\frac{c^2}{r^2}\frac{dr}{d\phi} = \frac{dy}{d\phi} = r \sin \phi,$$

$$\text{and } \therefore \frac{c^2}{2r^2} = \cos\phi - \cos\alpha,$$

if α be the deflection at B,

or
$$\sqrt{2}\,\frac{ds}{d\phi} = \frac{c}{\sqrt{\cos\phi - \cos\alpha}},$$

the intrinsic equation.

Putting $$\sin\frac{\phi}{2}=\sin\frac{\alpha}{2}\sin\psi,$$
this becomes
$$\frac{s}{c}=\int_0^\psi\frac{d\psi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\psi}}=F\left(\psi,\sin\frac{\alpha}{2}\right).$$

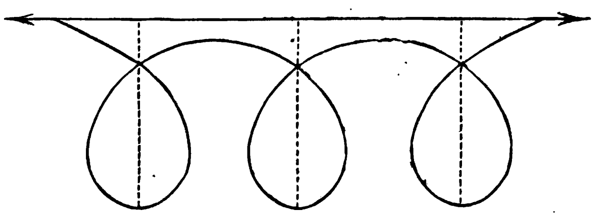
Hence the depth PL
$$=h-y=\frac{c^2}{r}$$
$$=c\sqrt{2}\sqrt{\cos\phi-\cos\alpha}=2c\sin\frac{\alpha}{2}\cos\psi$$
$$=2c\sin\frac{\alpha}{2}\,cn\frac{s}{c},\ \text{mod.}\sin\frac{\alpha}{2}.$$

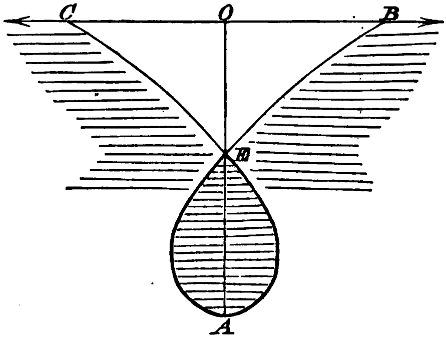
116. The Elastica is the curve formed by an elastic rod when bent, and is identical with the Lintearia.

Taking BAC as the rod, suppose the equilibrium maintained by forces at B and C in opposite directions.

The bending moment at P is proportional to the curvature (see Poisson's *Mécanique* or Minchin's *Statics*), and therefore, considering the equilibrium of the portion BP, and taking moments about P, it follows that the curvature at P varies as PL, so that
$$r\,.\,PL=c^2,$$
and the elastica is therefore identical with the lintearia.

117. The elastica may have any number of convolutions, as in the appended figures,





and the lintearia can be made to have convolutions by a proper adjustment of the water level and the water pressure.

Thus, if we imagine BC to be the water surface, and if arrangements be made to let the water fill the space AE, and press upwards on the portions BE, CE, we have a lintearia identical with an elastica of one convolution.

If we imagine that BC touches the bent rod at B and C, necessitating, as will be seen, an infinite length of rod, and if, as before, we measure the deflection from the tangent at A,

$$r = \infty\text{, when } \phi = \pi\text{, and therefore}$$

$$\frac{c^2}{2r^2} = 1 + \cos\phi, \text{ or } \frac{ds}{d\phi} = \frac{c}{2\cos\dfrac{\phi}{2}}.$$

Measuring s from A, this leads to

$$s = c \log \tan\left(\frac{\pi}{4} + \frac{\phi}{4}\right).$$

It will be seen hereafter that this is the Capillary curve.

118. To obtain the Cartesian equation of the Lintearia, we have

$$\frac{h-y}{c^2} = \frac{1}{r} = \frac{-\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \frac{-p\dfrac{dp}{dy}}{(1+p^2)^{\frac{3}{2}}},$$

$$\therefore \quad \frac{2hy - y^2}{2c^2} = \frac{1}{\sqrt{1+p^2}},$$

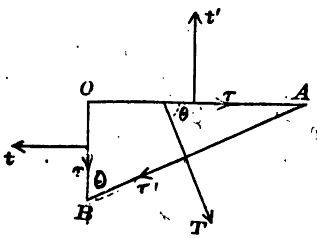
and
$$\frac{dx}{dy} = \frac{2hy - y^2}{\sqrt{4c^4 - (2hy - y^2)^2}}.$$

The investigation of the equation of the Lintearia was first effected by James Bernoulli.

119. Considering the equilibrium of a plane flexible membrane, the stress along any line, that is, the action between the contiguous portions of the surface bounded by that line, is in general oblique to the line, and is therefore represented by a tension t and a tangential action τ; we shall now shew that for any two directions, at right angles to each other, τ is the same, and that there are two directions for which τ vanishes.

Taking any small square element of the surface, the tangential actions $\tau\delta s$ and $(\tau + \delta\tau)\,\delta s$ on a pair of opposite sides form ultimately a couple $\tau\delta s^2$, if δs be a side of the element; and, since this must be balanced by the other couple, $\tau'\delta s^2$, if τ' be the tangential action in the direction at right angles, it follows that τ and τ' are equal.

Now take a small triangular element, OAB, right-angled at O, and represent the stresses as in the figure.



Resolving parallel to AB, we obtain

$$\tau' AB + tOB \sin\theta + \tau OB \cos\theta = t' OA \cos\theta + \tau OA \sin\theta,$$
$$\therefore \; 2\tau' = (t' - t) \sin 2\theta - 2\tau \cos 2\theta,$$

and τ' vanishes when $\tan 2\theta = \dfrac{2\tau}{t' - t}$, giving two directions at right angles.

120. If in the previous figure we assume that OA and OB are the directions of zero tangential action, and if we resolve in the directions perpendicular and parallel to AB, we shall obtain

$$T = t\cos^2\theta + t'\sin^2\theta,$$

$$\tau' = (t' - t)\sin\theta\cos\theta.$$

The quantities t and t' are now the greatest and least, or the least and greatest tensions, and we shall therefore call them the Principal Tensions.

121. If ϕ be the inclination of AB to OA produced, and if ϕ' be the inclination to OA of the resultant stress, $R \,.\, AB$, upon AB,

$$\tan\phi' = \frac{t'OA}{tOB} = \frac{t'}{t}\cot\theta,$$

$$\therefore\ \tan\phi'\tan\phi = -\frac{t'}{t},$$

also

$$R^2AB^2 = t^2OB^2 + t'^2OA^2,$$

$$\therefore\ R^2 = t^2\sin^2\phi + t'^2\cos^2\phi.$$

If then we describe an ellipse having its axes in the directions OA and OB, and such that

$$t' = \frac{1}{a} \text{ and } t = \frac{1}{b},$$

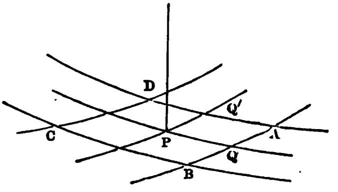
$$R^2 = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2},$$

and it therefore follows that if any line OP be drawn through O, the rate of stress upon it is represented by the radius vector of the ellipse, and the direction of the stress is that of the diameter conjugate to OP.

122. If now we take the case of a flexible membrane exposed to fluid pressure, and consider the equilibrium of a small element of the membrane, the results of the three preceding articles are at once applicable to the case, for in the limit the components of normal pressure disappear in comparison with the tangential action.

123. *A flexible surface of any form is exposed to the action of fluid; required to find the relation between the pressure, principal tensions, and the curvatures in the directions of these tensions, at any point*.*

Let Q, Q', be points contiguous to P, on the lines of principal tension PQ, PQ', through P; draw normal planes through Q and Q', perpendicular to the lines, PQ, PQ', cutting the surface in the arcs, AB, AD, and let BC, CD, be the arcs of section made by normal planes through contiguous points in $Q'P$, QP, produced.



The element BD is kept at rest by the tangential forces tAB, tCD, $t'AD$, $t'BC$, and the normal force, $p \,.\, AB \,.\, BC$.

Let r, r', be the radii of curvature at P of the curves PQ, PQ'; then, resolving along the normal at P, we have ultimately

$$p \,.\, AB \,.\, BC = 2tAB\frac{\frac{1}{2}AD}{r} + 2t'BC\frac{\frac{1}{2}AB}{r'},$$

and $$\therefore\; p = \frac{t}{r} + \frac{t'}{r'}.$$

If the nature of the surface be such that $t' = t$, the above equation is

$$\frac{p}{t} = \frac{1}{r} + \frac{1}{r'},$$

* The student must be guarded against the idea that there is any connection between principal tensions and principal curvatures.

For instance, imagine a membrane folded round a cylinder, and draw a number of helical lines of the same pitch on the membrane.

The membrane can be tightened in the directions of these lines, which will become the directions of greatest tension, the perpendicular tension being zero, and the stress along a generating line being oblique to that line.

or, if $z=f(x, y)$ be the equation to the surface,

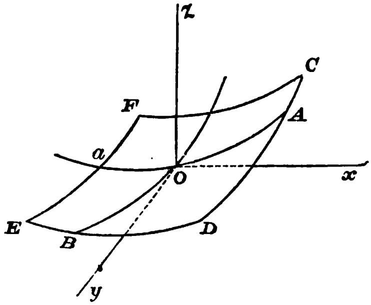
$$\frac{p}{t}\cdot\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}^{\frac{3}{2}}$$
$$=\left\{1+\left(\frac{dz}{dy}\right)^2\right\}\frac{d^2z}{dx^2}-2\frac{dz}{dx}\frac{dz}{dy}\frac{d^2z}{dxdy}+\left\{1+\left(\frac{dz}{dx}\right)^2\right\}\frac{d^2z}{dy^2};$$

the equation obtained by Lagrange and Poisson.

124. If the directions of t and t' are not those of principal tensions the tangential action will appear in the equation.

Taking any point O on the surface, two directions OA, OB at right angles to each other, let t, t' be the tensions in these directions, and T', T the tangential actions in the same directions.

Oz being the normal at O, draw four planes parallel to, and very near to, the normal planes AOz, BOz, cutting the surface in CD, DE, EF, FC.



Then, ultimately, the tangential actions, $T.CD$ and $T.EF$ on CD and EF are equal and opposite, as are also those on ED and CF.

Hence, by taking moments about OZ, it appears that $T=T'$, as in Art. 119.

If θ be the inclination to the plane xy of the tangent at A to the curve CD,

$$\tan\theta = \frac{d^2z}{dxdy} . OA,$$

and similarly at the point a,

$$\tan\theta' = \frac{d^2z}{dxdy}(-Oa).$$

Hence the sum of the actions $T . CD$ and $T . EF$ in the direction Oz

$$= T . CD \frac{d^2z}{dxdy} OA - T . EF \frac{d^2z}{dxdy}(-Oa) = T . CD . DE . \frac{d^2z}{dxdy},$$

and a similar term arises from the action T'.

Resolving along Oz, we now obtain

$$p . CD . DE = 2tCD \frac{OA}{r} + 2t'DE \frac{OB}{r'} + 2T . CD . DE \frac{d^2z}{dxdy},$$

and $$\therefore p = \frac{t}{r} + \frac{t'}{r'} + 2T \frac{d^2z}{dxdy} \text{*}.$$

125. If we imagine a surface of such a nature that the tension at any point is always perpendicular to a line of division through that point, it can be shewn that the tension at any point is the same in every direction.

Let a small triangular portion of the surface be supposed rigid; then the equilibrium in the tangent plane is entirely determined by the tensions of the sides of the triangle, for the tangential impressed forces, if there be any, will ultimately vanish in comparison with the tensions; and since these tensions are perpendicular to the sides, they must be in the ratio of their lengths, and therefore the measures of tension in all directions are the same.

Further, the tension will be the same over the surface, for, if a small rectangular element be considered, the tensions on the opposite sides must be equal.

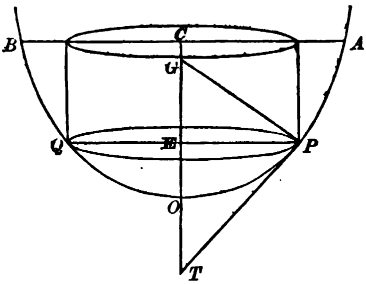
* The general question of the equilibrium of flexible surfaces is discussed in a paper, by myself, in the *Quarterly Journal of Mathematics*, Vol. IV. 1860.

The conception of such a surface is of the same nature as the conception of a perfectly rigid body or of a perfect fluid; nevertheless we obtain approximate specimens in the case of liquid films, such as soap-bubbles, or the films which may be seen in a clear glass bottle containing liquid which has been shaken about.

The consideration of the equilibrium of liquid films we defer to the next chapter.

126. *A vessel, formed of flexible and inextensible material, is in the form of a surface of revolution, and is held with its axis vertical, and filled with homogeneous liquid: it is required to determine the principal tensions at any point.*

Let O be the lowest point of the vessel, and take O for the origin.



Measure x vertically upwards, and let PEQ be any horizontal section, the upper rim being ACB, which is supposed to be fixed.

At all points of the horizontal section PQ, the tensions are evidently the same.

Let t be the meridional tension, i.e. the tension at P, in direction of the tangent at P to the curve AP, and t' the horizontal tension at P; these are the principal tensions.

The vertical resultant of the tension t along the section PQ counterbalances the resultant vertical pressure on the surface POQ; hence, if

$$OE = x, \; EP = y, \text{ and angle } PTO = \theta,$$

$$2\pi yt\cos\theta = \int_0^x g\rho\pi y'^2\,dx' + g\rho\pi y^2(c-x), \text{ if } OC = c.$$

This equation determines t, and t' is given by the equation

$$\frac{t}{r} + \frac{t'}{r'} = p, \text{ Art. } 123^*,$$

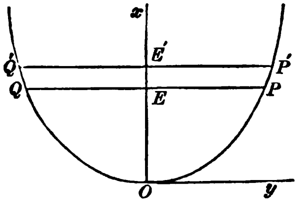
where $p = g\rho\,(c - z)$.

It will be observed that r is the radius of curvature of the curve AP at P, and that r', the radius of curvature of the perpendicular normal section, is the normal PG.

127. A more general proposition is the following:

A flexible vessel, in the form of a surface of revolution, is subject to fluid pressure, such that it is the same at all points of the same circular section; it is required to determine the principal tensions at any point.

Let PEQ, $P'E'Q'$ be two consecutive circular sections, and let t be the meridional tension at P.



If $OP = s$, the resultant tension, parallel to the axis, on the circle PQ,

$$= 2\pi yt\frac{dx}{ds};$$

$\therefore$ the resultant tension, parallel to Ox, on $P'Q'$

$$= 2\pi\left\{yt\frac{dx}{ds} + \frac{d}{ds}\left(yt\frac{dx}{ds}\right)\delta s\right\}, \text{ if } PP' = \delta s.$$

* This equation may also be obtained, for this case, by taking a small element bounded by lines of curvature, that is by meridians and horizontal circles; it will be necessary to employ Meunier's theorem, and to observe that the osculating planes of lines of curvature are not generally normal planes.

The difference of these two counterbalances the resultant pressure, parallel to Ox, on the strip of surface between the circles PQ, $P'Q'$, which is equal to

$$p \,.\, 2\pi y \delta s \frac{dy}{ds},$$

if p be the pressure at any point of the circle PQ;

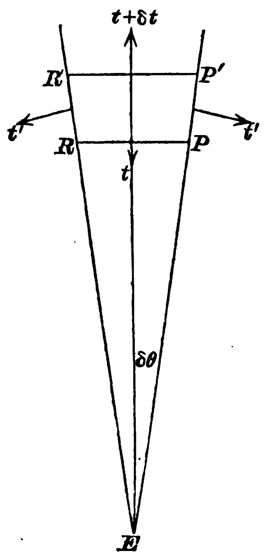
$$\therefore \frac{d}{ds}\left(yt\frac{dx}{ds}\right) = py\frac{dy}{ds},$$

and p being a given function of z, and therefore of s, this equation determines the tension t, and, as before, t' is given by the equation

$$\frac{t}{r} + \frac{t'}{r'} = p.$$

128. By eliminating p we obtain a relation between t and t', but it is better to obtain the relation directly.

Taking a small element PP', $R'R$ bounded by meridian arcs, PP', RR', and by circular arcs PR, $P'R'$, let $\delta\phi$ be the



angle between the meridian planes and $2\delta\theta$ the angle between the tangent lines, at P and R, to the meridians.

Then $$PR = y\delta\phi,\ PP' = \delta s,$$

and, resolving parallel to the direction of the meridian bisecting PR and $P'R'$,

$$\frac{d}{dy}(ty\delta\phi)\,\delta y = 2t'\delta s \sin\delta\theta$$

$$= t'\,\delta s \,.\, \frac{PR}{PE};$$

$$\therefore\ \frac{d}{dy}(ty) = t'\frac{ds}{dy}\cdot\frac{y}{PE} = t';$$

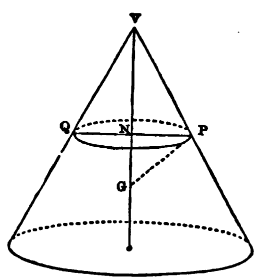
this equation, and, since $r' = y\sec\theta$,

the equation $$\frac{t}{r} + \frac{t'\cos\theta}{y} = p,$$

determine t and t'.

129. EXAMPLES. (1) *A conical perfectly flexible and elastic bag attached, mouth downwards, by the rim to a horizontal plane, and filled with liquid by a small hole at the apex, has, when at rest, the figure of a right circular cone; find the equation to the figure it will assume when detached and the liquid let out, neglecting its weight.*

Let t be the tension at P in the direction perpendicular to the generating line VP, t' the tension in the direction VP, and 2α the vertical angle of the cone.



Then $\qquad p = \frac{t}{r} + \frac{t'}{r'}$, gives, if $VN = x$,

$$g\rho x = \frac{t}{PG} = \frac{t}{x \tan \alpha \sec \alpha},$$

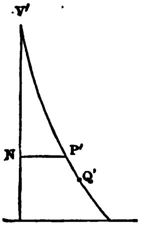
or
$$t = g\rho x^2 \tan \alpha \sec \alpha.$$

But $2\pi PNt' \cos \alpha =$ the resultant vertical pressure on VPQ

$$= \tfrac{2}{3}\, g\rho\pi x^3 \tan^2 \alpha;$$

$$\therefore\ t' = \tfrac{1}{3}\, g\rho x^2 \tan \alpha \sec \alpha.$$

Let $V'P'Q'$ be the generating curve of the surface of revolution into which the surface forms itself after the liquid has been let out, $V'N = \xi$, $P'N = \eta$, P' corresponding to the point P.



If $P'Q' = \delta s$, a small arc of the curve,

$$\delta x \sec \alpha = \delta s \left(1 + \frac{t'}{\lambda'}\right),$$

and
$$x \tan \alpha = \eta \left(1 + \frac{t}{\lambda}\right),$$

taking the modulus of elasticity different in the two directions. Taking account of the values of t and t' obtained above, x can be eliminated between these two equations, and the relation between ξ and η will result.

From the first equation, putting $\dfrac{g\rho \tan \alpha \sec \alpha}{3\lambda'} = \dfrac{1}{a^2}$,

$$\frac{ds}{dx} \cos \alpha = \frac{1}{1 + \dfrac{x^2}{a^2}};$$

$$\therefore\ \frac{s}{a} \cos \alpha = \tan^{-1} \frac{x}{a}, \text{ measuring } s \text{ from } V,$$

or
$$\frac{x}{a} = \tan\left(\frac{s}{a} \cos \alpha\right).$$

Substituting this expression for x in the second equation, we obtain

$$a \tan \alpha \tan\left(\frac{s}{a}\cos\alpha\right) = \eta\left\{1 + \frac{g\rho a^2 \tan\alpha \sec\alpha}{\lambda}\tan^2\left(\frac{s}{a}\cos\alpha\right)\right\},$$

as the differential equation to the curve.

If $\lambda = \lambda'$, $a \tan\alpha = \eta\left\{\cot\left(\frac{s}{a}\cos\alpha\right) + 3\tan\left(\frac{s}{a}\cos\alpha\right)\right\}$.

(2) *A flexible membrane in the form of a catenoid, that is, of the surface generated by the revolution of a catenary about its directrix, has its ends fastened to two equal circular boards of radius a, and the excess p of the air pressure inside over the air pressure outside is given.*

In this case the curvatures are in opposite directions, and if PG be the normal at P, each radius of curvature is equal to PG, and the equations of equilibrium are

$$t' - t = p\,.\,PG, \text{ and } t' = \frac{d}{dy}(yt);$$

$$\text{and since } PG = \frac{y^2}{c}, \qquad c\frac{dt}{dy} = py,$$

$$\therefore\quad 2c(t - \tau) = p(y^2 - c^2),$$

τ being the meridian tension of the vertex;

and $$t' = \tau + \frac{p}{2c}(3y^2 - c^2).$$

The first of these equations may at once be obtained by considering the equilibrium of the portion AB, and then the value t' follows from the equation, $t' - t = pr$.

Neglecting the weights of the boards, and supposing the form of equilibrium to be maintained by the inside air pressure, we obtain

$$2\pi a\left\{\tau + \frac{p}{2c}(a^2 - c^2)\right\}\frac{c}{a} = p\pi a^2,$$

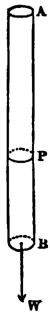
which gives $$2\tau = pc,$$

and the tensions then become,

$$t = \frac{py^2}{2c}, \text{ and } t' = \frac{3py^2}{2c}.$$

130. We have hitherto considered only laminae of uniform thickness, but, in order to include cases in which the lamina is of variable thickness, a more general measure of the tension can be given.

Suppose a bar AB of any homogeneous material to support a weight W, and let κ be the area of the section of the bar; then the tension at the section through P supports W and the weight of the bar PB; and if $\tau\kappa$ is equal to the sum of these weights, τ is the measure of the tension at P per unit of area.



It will be seen that τ is one dimension lower than t.

In fact, if e be the thickness of a flexible lamina at any point, the tension at which, measured in the usual way per unit of length of section, is t, we have

$$t\delta s = \tau e \delta s,$$

$$\text{or } t = e\tau.$$

131. The investigations of this chapter will not in general be applicable to surfaces which are inflexible, or of imperfect flexibility, but, if in any particular case the action between adjacent portions of a surface be wholly in the tangent plane, the relations obtained between the tension and the normal pressure will hold good.

For instance, if a vertical circular cylinder formed of any inflexible substance be filled with fluid, the action at any point will be wholly tangential and of the nature of tension.

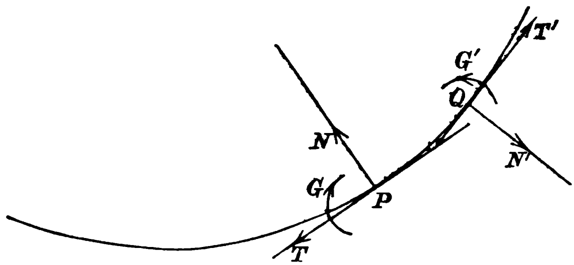
Rigid or elastic lamina subjected to fluid pressure.

132. We shall now consider the case of a cylindrical lamina, subjected to fluid pressure, such that it is the same along any generating line.

If APQ be a cross section perpendicular to the generating lines, the stress between the two portions separated by the generating line P will consist of a tangential force, a shearing force, and a couple.

Taking unit length of generating line, we shall denote these quantities by T, N, and G, observing that T, N, and G

represent the stresses exerted at P upon the element PQ, and that $T+\delta T$, $N+\delta N$, $G+\delta G$, in the contrary directions, are the actions at Q upon PQ.



If then $p\delta s$ be the fluid pressure upon PQ, we obtain by resolving parallel to the tangent and normal at P and by taking moments about P,

$$\delta T+(N+\delta N)\,\delta\phi+p\delta s\,\frac{d\phi}{2}=0,$$

$$\delta N-(T+\delta T)\,\delta\phi+p\delta s=0,$$

$$\delta G-(N+\delta N)\,\delta s-(T+\delta T)\,\frac{\delta s}{2}\,\delta\phi-p\delta s\,\frac{\delta s}{2}=0,$$

or, ultimately,

$$\frac{dT}{d\phi}+N=0,\qquad \frac{dN}{d\phi}-T+p\,\frac{ds}{d\phi}=0,$$

$$\frac{dG}{d\phi}-N\,\frac{ds}{d\phi}=0.$$

If the form of a rigid lamina be given, those equations determine the stress at any point.

If the lamina be elastic, we have the additional condition that G is proportional to the curvature, or that, $G=\frac{E}{r}$, and we can then eliminate G, N, and T, and obtain the differential equation of the form assumed by the lamina.

Further the equations will determine the law of fluid pressure under the action of which an elastic lamina will assume any assigned cylindrical form.

133. To illustrate the use of these equations, consider the case of an elliptic cylinder, formed of some thin rigid substance, closed at its ends and filled with air the pressure of which exceeds by p the pressure of the external air.

Eliminating N, we obtain

$$\frac{d^2T}{d\phi^2}+T=pr,$$

measuring s and ϕ from one end of the conjugate axis,

$$r=\frac{CD^3}{ab}=\frac{a^2b^2}{(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{3}{2}}},$$

and, by the method of the variation of parameters, it will be found that

$$T=p\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}+A\cos\phi+B\sin\phi,$$

and $\therefore$

$$N=A\sin\phi-B\cos\phi-p\frac{(a^2-b^2)\sin\phi\cos\phi}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}.$$

Employing the consideration of symmetry and the law of the equality of action and reaction, it follows that N vanishes at the apses and $\therefore$ $A=0$, and $B=0$.

Hence $\quad T=\dfrac{p\,.\,ab}{CD}$, and $N=-p\dfrac{a^2-b^2}{ab}\,CD\sin\phi\cos\phi$;

also $\quad \dfrac{dG}{d\phi}=Nr$ leads to

$$G=\frac{p}{2}\frac{a^2b^2}{a^2\sin^2\phi+b^2\cos^2\phi},$$

assuming that, when $p=0$, there is no stress.

134. As a further illustration, imagine a thin elastic lamina, naturally plane, pressed on one side by an excess p of air pressure, and supported at its side by two equal parallel forces.

In this case, $G=\dfrac{E}{r}$,

and
$$N=\frac{1}{r}\frac{dG}{d\phi}=-\frac{E}{r^3}\frac{dr}{d\phi};$$

$$\therefore\ \frac{dT}{d\phi}=\frac{E}{r^3}\frac{dr}{d\phi},\ \text{and}\ T=C-\frac{E}{2r^2};$$

Measuring from the middle point of the curve, and taking P for the force, and α for the deflection at each end, at which r is infinite,

$$C=P\sin\alpha.$$

$$\therefore\ T=P\sin\alpha-\frac{E}{2r^2};$$

$$\therefore\ \frac{d}{d\phi}\left(-\frac{E}{r^3}\frac{dr}{d\phi}\right)-P\sin\alpha+\frac{E}{2r^2}+pr=0,$$

$$\text{or}\ \frac{1}{r^3}\frac{d^2r}{d\phi^2}-\frac{3}{r^4}\left(\frac{dr}{d\phi}\right)^2+\frac{P\sin\alpha}{E}-\frac{1}{2r^2}-\frac{pr}{E}=0,$$

an equation which can be integrated by putting $z=\left(\frac{dr}{d\phi}\right)^2$, and we thus obtain $\frac{dr}{d\phi}$ in the terms of r.

EXAMPLES.

1. Supposing the cylinders of a Bramah's Press made of the same material, and the stress to be the same in each, what should be the ratio of the thicknesses of the cylinders?

2. A cylindrical vessel is formed of metal a inches thick, and a bar of this metal of which the section is A square inches, will just bear a weight W without breaking. If the cylinder be placed with its axis vertical, find how much fluid can be poured into it without bursting it.

3. The tensile strength of cast iron being 16000 lbs. weight per square inch of section, find the thickness of a cast iron water-pipe whose internal diameter is 12 inches, that the stress upon it may be only one-eighth of its ultimate strength when the head of water is 384 feet.

4. A hollow cone, the vertex of which is downwards, is filled with water; find where the horizontal tension is greatest.

Also find where the tension in the direction of a generating line is greatest.

5. The top of a rectangular box is closed by an uniform elastic band, fastened at two opposite sides, and fitting closely to the other sides; the air being gradually removed from the box, find the successive forms assumed by the elastic band, and when it just touches the bottom of the box, find the difference between the external and internal atmospheric pressures.

6. An elastic tube of circular bore is placed within a rigid tube of square bore which it exactly fits in its unstretched state, the tubes being of indefinite length; if there be no air between the tubes and air of any pressure be forced into the elastic tube, shew that this pressure is proportional to the ratio of the part of the elastic tube that is in contact with the rigid tube to the part that is curved. Π orig. pr. $p - \Pi$ = extra pr.

7. A vessel, formed of a thin substance, in the shape of a cone with its axis vertical and vertex downwards, is just filled with liquid and closed at the top. If it be made to rotate uniformly about its axis, find the principal tensions at any point.

8. A spherical elastic envelope is surrounded by, and full of, air at atmospheric pressure (Π), when an equal amount is forced into it. Prove that the tension at any point of the envelope then becomes $\frac{\Pi}{2r'^2}(2r^3 - r'^3)$, where r, r' denote the initial and final radii.

9. An elastic spherical envelope whose natural radius is a, has air forced into it so that its radius becomes b; it is then placed under an exhausted receiver, and its radius increases to c; find the quantity of air forced in, assuming that the tension is proportional to the increase of surface.

10. An elastic spherical envelope of radius a, is filled with air at the same pressure and temperature as the surrounding

air. Assuming that the tension varies as the increase of surface, and that if the quantity of air inside be doubled the radius becomes ma, and that if the temperature inside be then raised to T', the radius becomes na, prove that

$$2\frac{T'}{T}=n^3+\frac{n^4m^2(n^2-1)(2-m^3)}{m^2-1}.$$

11. A hemispherical bag, supported at its rim, is filled with water; the principal tensions at a depth x are in the ratio

$$x^2+ax+a^2 : 2x^2+2ax-a^2.$$

Find also where the horizontal tension vanishes, and explain the circumstance of its being negative for a portion of the bag.

12. If the hemispherical bag be closed at the top by a rigid plane to which its rim is tied, and then inverted, shew that the principal tensions at a depth x, are in the ratio

$$3a-2x : 9a-4x.$$

13. A spherical envelope is just filled with liquid, which rotates uniformly about a diameter; neglecting gravity, prove that the principal tensions at an angular distance ϕ from the axis of rotation are

$$\tfrac{1}{8}\rho\omega^2a^3\sin^2\phi \text{ and } \tfrac{3}{8}\rho\omega^2a^3\sin^2\phi.$$

14. A cylindrical shell of finite thickness is formed of a material such that a bar, one square inch in section, can sustain a tension τ without giving way. If this shell be subjected to an internal fluid pressure ϖ, which is only just not sufficient to burst the cylinder, prove that $\varpi=\tau\log\frac{a}{b}$; where a and b are the external and internal radii of the shell.

15. Shew, from the equations of Art. 127, that, if t be equal to t' at every point, each is constant.

Shew also that, in general, if t be a maximum or a minimum, it will be equal to t'.

16. A flexible bag, in the form of a right circular cone, just filled with liquid, has the rim of its base fastened to a rigid plane, and the liquid is acted upon by repulsive forces from the centre of the base, varying as the distance; find the principal tensions at any point.

If an aperture be made in the rigid plane, fitted with a piston, and a blow be struck on the piston, find the principal impulsive tensions at any point.

17. If, in Art. 127, the vessel be a paraboloid, and if the principal tensions be equal at any point of the horizontal section through the focus, shew that the length of the axis is $\frac{5}{8}$ths of the latus-rectum.

18. A convex inextensible pliable envelope in the form of a surface of revolution with its axis vertical is exposed to water-pressure from within. Prove that at the widest part the tension along the meridians is a maximum or minimum according as it is less or greater than the tension across the meridians.

19. A flexible bag is in the form of a surface of revolution and subjected to a constant fluid pressure. Find the form in order that the ratio of the pressure to the maximum principal tension may be twice the curvature; and find the principal tensions at any point.

20. A quantity of liquid within a thin spherical shell rotates about a vertical axis with uniform angular velocity: find the principal tensions at any point, and examine the effects of an increase in the velocity of rotation.

21. A flexible surface, such that the tension at any point is the same in every direction; and whose form is given by the equation $z = \phi(x, y)$, is exposed to the action of fluid, find the ratio of the pressure to the tension at any point.

Shew that this ratio is 11 : 12 at the points of the surface $4x^2 = 3z^2(x^2 + y^2)$, where $x = y = z$.

22. A right circular cylinder is made of elastic material attached to rigid fixed plane ends. It is distended by fluid

pressure. Supposing that the tensions in the meridian and circular sections are regulated by Hooke's law, obtain equations sufficient to determine completely the shape it will assume. If the pressure p be constant, prove that the meridian curve is

$$x + A = \int \frac{\frac{py^2}{2} + B}{\sqrt{\left(\frac{\lambda y^2}{2a} - \lambda y + C\right)^2 - \left(\frac{py^2}{2} + B\right)^2}} dy,$$

where a is the original radius, λ one of the moduli of elasticity, and A, B, C, constants of integration.

23. A plane elastic lamina, resting against two parallel bars, is bent by fluid pressure on one side into the form of a catenary; find the law of the fluid pressure.

24. A vessel of thin rigid material, in the form of half a circular cylinder is filled with water, and supported by vertical forces at its bounding generating lines which are horizontal; prove that the stresses at any point distant ϕ from the lowest point are such that

$$\frac{2T}{g\rho a^2} = \phi \sin \phi + \cos \phi, \quad \frac{2N}{g\rho a^2} = -\phi \cos \phi,$$

and
$$\frac{2G}{g\rho a^3} = \frac{\pi}{2} - \phi \sin \phi - \cos \phi.$$

25. A rigid lamina in the form of a cylinder the cross section of which is a catenary, is subjected to air pressure on the concave side and supported by two equal forces parallel to the axis of the catenary; find expressions for the stresses at any point.

26. A plane elastic lamina rests on two parallel horizontal bars, and is bent downwards between the bars by a constant air pressure above; form the differential equation connecting the radius of curvature and the deflection.

Also form the differential equation when the bending is effected by pouring water on the lamina to the level of the bars.

CHAPTER IX.

CAPILLARITY.

135. It is a well known fact that if a glass tube of small bore be dipped in water, the water inside the tube rises to a higher level than that of the water outside.

It is equally well-known that if the tube be dipped in mercury, the mercury inside is depressed to a lower level than that of the mercury outside.

If a glass tumbler contain water it will be seen that at the line of contact the surface is curved upwards and appears to cling to the glass at a definite angle.

If the tumbler be carefully filled, the level of the water will rise above the plane of the top of the tumbler, the water bulging over the round edge of the top.

If water be spilt on a table, it has a definite boundary, and the curved edges cling to the table.

These facts, and many others, have led to the theory of the existence of a surface tension, the laws of which may be stated as follows:

(1) *At the bounding surface separating air from a liquid, or between two liquids, there is a surface tension which is the same at every point and in every direction.*

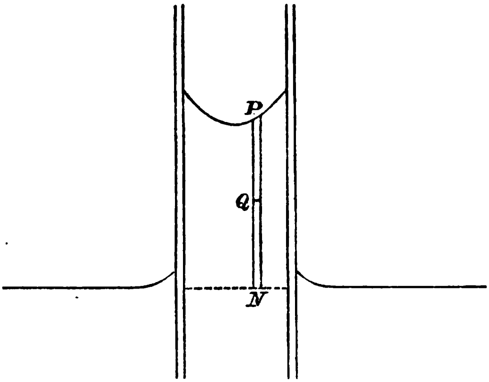
(2) *At the line of junction of the bounding surface of a gas and a liquid with a solid body, or of the bounding surface of two liquids with a solid body, the surface is inclined to the surface of the body at a definite angle, depending upon the nature of the solid and of the liquids.*

In the case of water in a glass vessel the angle is acute; in the case of mercury it is obtuse.

Assuming these laws we can account for many of the phenomena of capillarity and of liquid films.

136. *Rise of liquid between two plates.*

If t be the surface tension, α the constant angle at which the surface meets either plate, called the angle of capillarity, h the mean rise, and d the distance between the plates, we have, for the equilibrium of the unit breadth of the liquid,



$$2t \cos \alpha = g\rho hd,$$

so that the rise increases with the diminution of the distance between the plates.

It will be seen that the pressure at any point Q is less than the pressure at N by $g\rho \, . \, QN$.

$$\text{and } \therefore = \Pi - g\rho QN.$$

The atmospheric pressure at P being sensibly equal to the pressure at the water level outside, it follows that the weight PN is supported by the resultant of the surface tensions on its upper boundary.

137. *Rise of a liquid in a circular tube.*

In this case the column of liquid is supported by the tension round the periphery of its upper boundary, and therefore, if r be the internal radius,

$$2\pi rt \cos \alpha = g\rho \pi r^2 h,$$

or

$$2t \cos \alpha = g\rho rh.$$

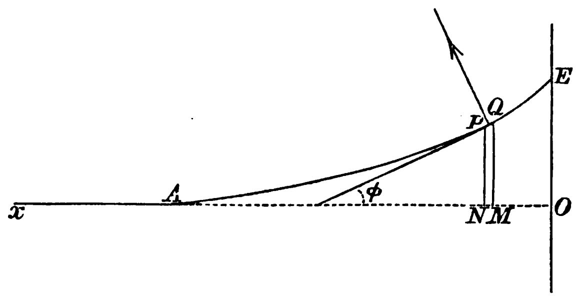
The pressure at any point of the suspended column being less than the atmospheric pressure, it follows that if the column were high enough, the pressure would merge into a state of tension, which would still follow the law of fluid pressure of being the same in every direction.

It may be observed that the potential energy, due to the ascent of the column, is independent of the radius.

The Capillary Curve.

138. The capillary curve is the form assumed by the surface of a liquid in contact with a vertical wall.

If OE be the vertical wall, and OA the natural surface of the liquid, consider the equilibrium of the elementary portion $PQMN$, taking one unit of breadth perpendicular to the plane of the paper.



If $PQ = \delta s$, the vertical resultant of the surface tensions at P and Q is

$$2t\frac{\frac{1}{2}\delta s}{r}\cos\phi,$$

r being the radius of curvature at P, and this supports the weight of the column, since the atmospheric pressure at P, and the liquid pressure at N are practically equal,

$$\therefore \frac{t\cos\phi}{r}\delta s = g\rho y\delta x,$$

and $$ry = \frac{c^2}{4}, \text{ if } 4t = g\rho c^2,$$

and, inverting the figure of Art. (117), we see that the capillary curve is a particular case of the elastica.

The particularity consists in the fact that OA is a tangent to the curve, so that $\frac{dy}{dx} = 0$ when $y = 0$, and enables us to obtain the Cartesian equation.

To do this, we have $$\frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \frac{4y}{c^2},$$

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{c^2}{c^2 - 2y^2} \text{ or } \frac{dx}{dy} = \frac{2y^2 - c^2}{2y\sqrt{c^2 - y^2}}.$$

Integrating, and taking the origin so that $y = c$, when $x = 0$,

we obtain $$x + \sqrt{c^2 - y^2} = \frac{c}{2} \log \frac{c + \sqrt{c^2 - y^2}}{y},$$

or $$\frac{y}{c} = \operatorname{sech}\left\{\frac{2}{c}(x + \sqrt{c^2 - y^2})\right\}.$$

If $y = 0$, x is infinite, and, taking the figure of Art. (117), the elastica is identical with the capillary curve when BC is the tangent at B and C, but this is only possible when the length is very great.

For the intrinsic equation in this case,

$$-\frac{c^2}{4r^2}\frac{dr}{d\phi} = \frac{dy}{d\phi} = r \sin \phi,$$

$$\therefore \frac{c^2}{8r^2} = 1 - \cos \phi, \qquad \frac{ds}{d\phi} = \frac{c}{4 \sin \frac{\phi}{2}},$$

and $\frac{2s}{c} = \log \frac{\tan \frac{\phi}{4}}{\tan \frac{\alpha}{4}}$, measuring s from the point $\phi = \alpha$.

139. If a drop of liquid be placed on a horizontal plane, it will in general take the form of a surface of revolution.

Measuring x vertically downwards from the highest point, the equation of equilibrium will be

$$\frac{1}{r}+\frac{1}{r'}=\frac{p}{t}=\frac{2}{a}+\frac{g\rho x}{t},$$

if a be the radius of curvature at the highest point.

If the drop be large so that we may consider the top flat, and if we neglect the curvature of horizontal sections,

$$\frac{1}{r}=\frac{x}{c^2},\ -\frac{1}{r^2}\frac{dr}{d\phi}=\frac{\cos\phi}{c^2}r,\ \text{and} \therefore r=\frac{c}{\sqrt{2\sin\phi}},\ \text{approximately},$$

or, in Cartesians,

$$\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}=\frac{x}{c^2},\quad \text{and}\quad \frac{\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}=\frac{x^2}{2c^2}+C.$$

If h be the depth below the top at which the tangent is vertical, this leads to $h^2=2c^2$, and gives the expression $\frac{1}{2}g\rho h^2$ for the surface tension.

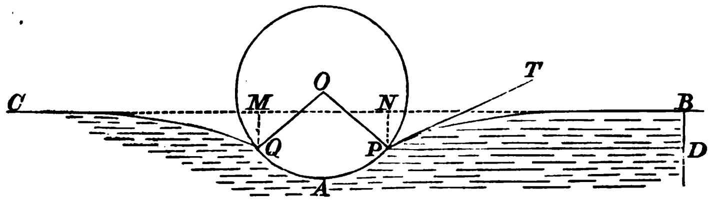
140. If water be introduced between two parallel plates of glass, so near to each other that the action of gravity may be neglected, the water pressure inside will be constant, and if the surface be a surface of revolution, it will follow the same law, and be subject to the equation

$$\frac{1}{r}+\frac{1}{r'}=\frac{p}{t},$$

p being the difference between the external and internal pressures.

141. The well known experiment of floating a needle on the surface of water can be explained by aid of the laws of surface tension.

The figure representing a section of the needle and the surface of the water at right angles to the axis of the needle,



the forces in action on the needle are the tensions at P and Q, and the water pressure on PAQ, which is equal to the weight of the volume $NPAQM$ of water; these forces counterbalance the weight of the needle.

Further the horizontal component of the tension at P, together with the horizontal water pressure on BD, is equal to the tension at B, PD being horizontal and BD vertical.

These conditions determine the equilibrium, and lead to the equations,

$$2t \sin \overline{\theta - \alpha} + g\rho c\,(c\theta + c \sin \theta \cos \theta - 2h \sin \theta) = w$$

$$4t \sin^2 \frac{\theta - \alpha}{2} = g\rho\,(c \cos \theta - h)^2,$$

where α is the angle of capillarity, w the weight of the needle, h the height of its axis above the natural level of the water, and 2θ the angle POQ.

Liquid Films.

142. Liquid films are produced in various ways; a soap-bubble is a familiar instance, and liquid films may be formed, and their characteristics observed, by shaking a clear glass bottle containing some viscous liquid, or by dipping a wire frame into a solution of soap and water, or glycerine, and slowly drawing it out.

The fact that films apparently plane, can be obtained shews that the action of gravity may be neglected in comparison with the tension of the film.

It is found that a very small tangential action will tear the film, and it is therefore inferred that the stress across any

line is entirely normal to that line. From this it follows, as in Art. (125), that the tension is the same in every direction.

143. *Energy of a plane film.*

If a plane film be drawn out from a reservoir of viscous liquid, a certain amount of work is expended, and the work thus expended represents the potential energy of the film.

Imagine a rectangular film $ABCD$, bounded by straight wires AD, BC, AB being in the surface of the liquid, and CD a moveable wire.

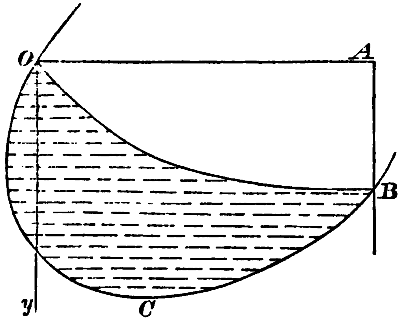
The work done in pulling out the film is equal to $t.AB.AD$, and therefore, if S be the superficial energy, per unit of area, it follows that

$$S = t.$$

It should be observed that what we have here called the tension of the film is equal to twice the surface tension of either side of the film.

144. *A wire in a vertical plane of any shape has a piece of thread, of given length and weight, fastened at two points, and the wire and the thread form the boundary of a plane liquid film.*

To find the form assumed by the thread, we shall express the condition that the potential energy of the system is a minimum.



If A be the area $OABC$, the energy of the film

$$= SA - \int Sydx,$$

and therefore if w be the weight of unit length of thread, the potential energy of the system is a minimum when

$$\int S y dx + w \int y ds$$

is a maximum,
with the condition

$$\int ds = l.$$

By the aid of the calculus of variations, this leads to an equation of the form,

$$\frac{dx}{dy} = \frac{y}{\sqrt{ay^2 + by + c}},$$

which is easily integrated.

This equation may represent, for certain values of the constants, a circle or a catenary, as is obvious *a priori*.

This question might otherwise have been solved by writing down the conditions of equilibrium of an element of the thread.

145. *Energy of a spherical soap bubble.*

If p be the difference between the internal and external pressures, $2t = pr$, and, assuming t constant, the work done in expanding from r to $r + \delta r$, is

$$p \,.\, 4\pi r^2 \delta r, \text{ or } 8\pi t r \delta r;$$

$\therefore$ the total work done in forming a bubble of radius $c = 4\pi t c^2$, and as before the superficial energy $= t$.

And, in general, whatever be the form of a film, its energy $= t \,.\, S$, if S be the surface, for the energy of a plane element $= t\delta S$.

The Forms of liquid films.

146. If the air pressure be the same on both sides of a film, the condition of equilibrium is that

$$\frac{1}{r} + \frac{1}{r'} = 0,$$

or that the mean curvature is zero.

This condition is satisfied in the cases of the catenoid and the helicoid, which are therefore possible forms of liquid films.

In Cartesian co-ordinates the equation becomes

$$\left\{1+\left(\frac{dz}{dy}\right)^2\right\}\frac{d^2z}{dx^2}-2\frac{dz}{dx}\frac{dz}{dy}\frac{d^2z}{dxdy}+\left\{1+\left(\frac{dz}{dx}\right)^2\right\}\frac{d^2z}{dy^2}=0.$$

as in Art. (123).

The discussion of this equation is the subject of many memoirs by eminent mathematicians, and several very remarkable special solutions have been obtained.

For instance the surfaces,

$$\epsilon^z=\frac{\cos y}{\cos x}, \quad \text{and} \quad 4\sin z=(\epsilon^x-\epsilon^{-x})(\epsilon^y-\epsilon^{-y}),$$

will be each found to possess the property that its mean curvature is zero *.

In Plateau's work, *Sur les liquides soumis aux seules forces moleculaires* (2 vols. 1873), will be found an elaborate account of the labours of mathematicians on this subject, and of his own extensive series of experiments, but, within the limits of this treatise, we are unable to do more than to call attention to the work of Plateau and others, and to place before the student the leading ideas and some of the most important results.

147. If the form of the film be that of a surface of revolution, then taking the axis of the surface as the axis of z,

$$r^2=x^2+y^2=f(z).$$

Substituting in the equation above, we obtain

$$\{f'(z)\}^2+2r^2-r^2f''(z)=0;$$

which, by transformation, becomes

$$r\frac{d^2r}{dz^2}-\left(\frac{dr}{dz}\right)^2=1.$$

Integrating,

$$\frac{dz}{dr}=\frac{a}{\sqrt{r^2-a^2}}, \text{ and } \therefore\ z+b=a\log\left(r+\sqrt{r^2-a^2}\right),$$

* Catalan, *Journal de l'École Polytechnique*, 1856.

or
$$2r = \epsilon^{\frac{z+b}{a}} + a^2\epsilon^{-\frac{z+b}{a}},$$
assuming
$$a^2\epsilon^{-\frac{2b}{a}} = \epsilon^{-\frac{2h}{a}},$$
the result is
$$2r = a\left(\epsilon^{\frac{z+h}{a}} + \epsilon^{-\frac{z+h}{a}}\right),$$
shewing that a catenoid is the only possible form of revolution of a film when the pressure is the same on both sides.

148. The same result is obtained by the principle of energy, for the surface
$$\int 2\pi y ds$$
is then a maximum or a minimum, and, by the calculus of variations, this leads to a catenary as the generating curve, the axis of revolution being the direction of the catenary.

Now it is well known that if the origin, the directrix, and one point of the curve be given, there are two catenaries which can be drawn, above and below the catenary of minimum tension at the given point.

These correspond to the maximum and minimum surfaces, the former being the form of unstable equilibrium for the film, and the latter of stable equilibrium, in accordance with the known law that positions of stable and unstable equilibrium occur alternately.

149. If the pressures on the two sides of a film be different, and if p be the difference, the condition of equilibrium is
$$\frac{1}{r} + \frac{1}{r'} = \frac{p}{t},$$
or that the mean curvature is constant.

We shall apply the principle of energy to prove this relation for the case of surfaces of revolution.

The fact that p is constant may be expressed by closing the ends and assuming that the volume of air inside is constant.

The variation of the expression

$$\int(2\pi y ds + \lambda\pi y^2 dx)$$

is therefore zero.

This leads to

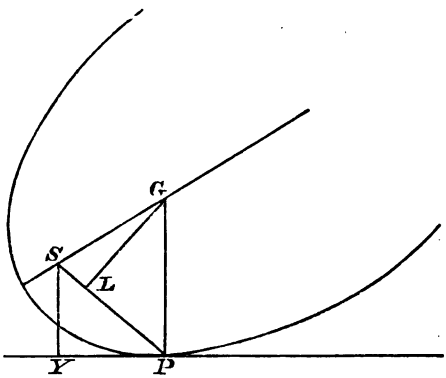
$$\frac{dx}{ds} = \frac{c}{y} - \frac{\lambda y}{2}, \text{ and } \therefore \frac{d^2x}{ds^2} = \left(-\frac{c}{y^2} - \frac{\lambda}{2}\right)\frac{dy}{ds}.$$

Hence, if PG be the normal,

$$\frac{1}{PG} \pm \frac{1}{r} = -\lambda, \text{ since } \frac{d^2x}{ds^2} = \mp\frac{1}{r}\frac{dy}{ds},$$

according as the curve is convex or concave to the axis of x; that is the mean curvature is constant. And, in the general case, we have to express the condition that the surface is a maximum or a minimum with a given volume, leading to the same general result*.

150. If the film be in the form of a surface of revolution, we can shew that the meridian curve is the path of the focus of a conic rolling on a straight line.



If ρ be the radius of curvature of the conic, and r the radius of curvature of the path of S,

* See Jellet's *Calculus of Variations*, or Todhunter's *Integral Calculus*.

$$\frac{1}{r} = \frac{1}{SP} - \frac{\rho \cos SPG}{SP^2}\text{*},$$

$$= \frac{1}{SP} - \frac{PG^2}{PL \,.\, SP^2}, \text{ } GL \text{ being perpendicular to } SP,$$

$$= \frac{1}{SP} - \frac{PL}{SY^2},$$

$$\therefore \frac{1}{r} + \frac{1}{SP} = \frac{2}{SP} - \frac{PL}{SY^2}.$$

In the case of the parabola, this vanishes, and $r = -SP$.

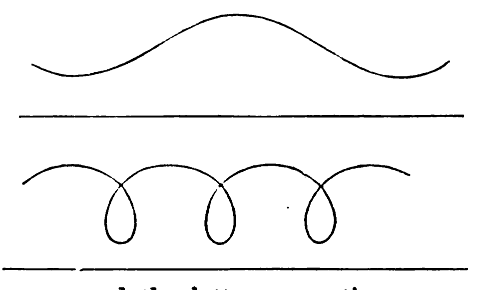
For the ellipse,

$$\frac{SY^2}{SP^2} = \frac{BC^2}{SP \,.\, HP}, \text{ and } \frac{1}{r} + \frac{1}{SP} = \frac{1}{AC},$$

and for the hyperbola,

$$\frac{1}{r} + \frac{1}{SP} = -\frac{1}{AC}.$$

The first is the catenoid; the second and third are called by Plateau the Unduloid, and the Nodoid, the former being



a sinuous curve, and the latter presenting a succession of nodes.

To obtain a clear view of the generation of the nodoid, it must be considered that, as one branch of the hyperbola rolls, the point of contact moves off to an infinite distance; the

* See *Roulettes and Glissettes.*

line then becomes asymptotic to both branches, and the other branch begins to roll, thereby producing a perfect continuity of the figure*.

Of the numerous works and papers on the subject of liquid films the student will find full accounts in Plateau's work, and in Professor Clerk Maxwell's article in the British Cyclopædia.

EXAMPLES.

1. Two spherical soap bubbles are blown, one from water, and the other from a mixture of water and alcohol: if the tensions per linear inch are equal to the weights of one grain and $\frac{7}{12}$ grain respectively, and if the radii be $\frac{7}{8}$ inch and $1\frac{1}{3}$ inch respectively, compare the excess, in the two cases, of the total internal over the total external pressure.

2. If two soap bubbles are blown from the same liquid, of radii r and r', and if the two coalesce into a single bubble of radius R, prove that, if Π be the atmospheric pressure, the tension is equal to

$$\frac{\Pi}{2}\cdot\frac{R^3-r^3-r'^3}{r^2+r'^2+R^2}.$$

If a soap bubble be placed between, and in contact with, two parallel plates, which are then slowly drawn apart, what are the forms, synclastic and anticlastic, which can be assumed?

3. The superficial tensions of the surfaces separating water and air being 8·25, water and mercury 42·6, mercury and air 55, what will be the effect of placing a drop of water upon a surface of mercury?

4. A drop of oil, placed on the surface of water, at once spreads itself out into a layer of extreme tenuity; explain the cause of this expansion of the oil, and prove, from observation of an attendant phenomenon, that the thickness of the layer may become less than ·00001 of an inch.

* (Plateau, Vol. I. p. 136. See also an article by Delaunay, *Liouville's Journal*, 1841, and an article by Lamarle, *Bulletins de l'Académie Belgique*, 1857.)

What will take place if another drop of oil is placed on the surface?

5. Shew that if a light thread with its ends tied together form part of the internal boundary of a liquid film, the curvature of the thread at every point will be constant.

If the thread have weight, and if the film be a surface of revolution about a vertical axis, prove that, in the position of equilibrium, the tension of the thread is

$$\frac{l}{2\pi}\sqrt{\tau^2 - w^2},$$

l being its length, w its weight per unit length, and τ the tension of the film.

6. A plane liquid film is drawn out from a soap-sud reservoir; prove that the numerical value of the energy per unit of area (e) is equal to that of the tension (T) per unit of length.

If the film be removed from the reservoir, and if σ denote subsequently the mass of unit of area, prove that

$$T = e - \sigma \frac{de}{d\sigma}.$$

7. Any number of soap-bubbles are blown from the same liquid and then allowed to combine with one another. Find the radius of the resulting bubble, and prove that the decrease of surface bears a constant ratio to the increase of volume.

8. The surface tension of water exposed to air is such that the stress across an inch is equal to the weight of about 3·3 grains. If 1,000,000,000 spherical drops combine to form a single spherical rain-drop $\frac{1}{10}$ inch in diameter, shew that the work done by the surface tensions is equal to about ·0001277 foot-pounds.

internal

9. If a film under unequal and external pressure form a surface of revolution, prove that the inclination ϕ of the tangent plane at P to the axis is given by the equation

$$\cos\phi = \frac{x}{a} + \frac{b}{x}:$$

x being the perpendicular from P on the axis and a, b constants.

10. A drop of liquid with uniform surface-tension is made to revolve about an axis. Prove that the meridian curve of the surface will be the roulette of the pole of the curve

$$\frac{p^2}{c^2} = \frac{2a}{r} - 1.$$

11. Two soap-bubbles are in contact; if r_1, r_2, be the radii of the outer surfaces, and r the radius of the circle in which the three surfaces intersect,

$$\frac{3}{4r^2} = \frac{1}{r_1^{\,2}} + \frac{1}{r_2^{\,2}} - \frac{1}{r_1 r_2}.$$

12. If a frame of fine straight wire in the form of a tetrahedron be lowered into a solution of soap and water and drawn up again, there are found in certain cases plane films starting from the edges and meeting in a point. Shew that this is not a possible form of equilibrium for every tetrahedron, and that it is so if one face be an equilateral triangle and the others isosceles triangles, whose vertical angles are each less than $\sec^{-1}(-3)$.

13. If water be introduced between two parallel plates of glass, at a very small distance d from each other, prove that the plates are pulled together with a force equal to

$$\frac{2At\cos a}{d} + Bt\sin a,$$

A being the area of the film and B its periphery.

14. A hollow right circular cone of glass is placed with its axis vertical and vertex upwards in homogeneous liquid. Find the height to which the liquid will be raised in the cone, and write down the differential equation of the surface inside. Deduce results for a cylinder.

15. A needle floats on water with its axis in the natural level of the surface; if σ be the specific gravity of steel referred to water, β the angle of capillarity, and 2α the angle

subtended at the axis by the arc of a cross-section in contact with the water, prove that

$$(\pi\sigma - a)\sin\tfrac{1}{2}(a-\beta) = \cos a \cos\tfrac{1}{2}(a+\beta).$$

16. A film of fluid adheres to two wires each of which forms one turn of a helix, the axes of the two helices being coincident, and their steps equal. Shew that the condition of equilibrium of the film will be satisfied if the differential equation to any section of the film through the axis is of the form

$$dx = \frac{A\,dy}{y}\sqrt{\frac{\alpha^2 + y^2}{y^2 - A^2}},$$

when $2\pi\alpha =$ step of either helix: (i.e. distance between consecutive threads).

17. To the extremities of the axis of a wire helix of pitch b, whose length is very great compared with its diameter, an elastic string (modulus of elasticity E) is fastened, the wire being bent over radially at each end so as to meet the axis. The string when straight is tight but unstretched. If the helix and string be dipped into a solution of soap and then removed with a film adhering to the wire and string, shew that, except near the ends, the string will be drawn into a helix of radius r where r is given by the equation

$$(16\pi^4h^2T^2 - 64\pi^6E^2)\,r^4 + 32\pi^4h^2TEr^3 + 8\pi^2h^4T^2r^2$$
$$+ 8\pi^2h^4TEr + h^6T^2 = 0,$$

T representing the whole tension per unit of length (of both surfaces) of a soap-film.

CHAPTER X.

THE EQUILIBRIUM OF REVOLVING LIQUID, THE PARTICLES OF WHICH ARE MUTUALLY ATTRACTIVE.

151. If a liquid mass, the particles of which attract each other according to a definite law, revolve uniformly about a fixed axis, it is conceivable that, for a certain form of the free surface, the liquid particles may be in a state of relative equilibrium; since, however the resultant attraction of the mass upon any particle depends in general upon its form, which is unknown, a complete solution of the problem cannot be obtained.

For any arbitrarily assigned law of attraction, the question is one of purely abstract interest, and it is only when the law is that of gravitation that it becomes of importance, from its relation to one of the problems of physical astronomy.

We shall consider the fluid homogeneous, and confine our attention to two cases; in the first of these the attractive forces are supposed to vary directly as the distance, and, in the second, to follow the Newtonian law.

152. *A homogeneous liquid mass, the particles of which attract each other with a force varying directly as the distance, rotates uniformly about an axis through its centre of gravity; required to determine the form of the free surface.*

The resultant attraction on any particle is in the direction of, and proportional to, the distance of the particle from the centre of gravity; and if μ be a measure of the whole mass of fluid, μx, μy, μz, may represent the components of the attraction, parallel to the axis, on a particle of fluid about the point x, y, z.

Taking the origin at the centre of gravity, and axis of rotation as the axis of z, the equation of equilibrium is

$$dp = \rho \{(\omega^2 x - \mu x)\, dx + (\omega^2 y - \mu y)\, dy - \mu z dz\};$$

and therefore

$$p = C + \frac{\rho}{2} \{(\omega^2 - \mu)(x^2 + y^2) - \mu z^2\}.$$

At the free surface p is zero or constant, and the equation to the free surface is

$$\left(1 - \frac{\omega^2}{\mu}\right)(x^2 + y^2) + z^2 = D,$$

the constant D depending upon ω, and upon the mass of the fluid.

If $\omega^2 < \mu$, the free surface is a spheroid which becomes more oblate as ω increases, and when $\omega^2 = \mu$, the free surface consists of two planes; to render this possible we may conceive the fluid enclosed within a cylindrical surface, the axis of which coincides with the axis of rotation.

When $\omega^2 > \mu$, the free surface is a hyperboloid of two sheets, which for a certain value (ω') of ω becomes a cone, the fluid filling the space between the cone and the cylinder. Taking account of the volume of the fluid, the value of ω' can be determined by putting $D = 0$, since the pressure in this case vanishes at the origin.

If $\omega > \omega'$, the surface is a hyperboloid of one sheet, which, as ω increases, approximates to the form of a cylinder, and it is therefore necessary, for large values of ω, to conceive the containing cylinder closed at its ends.

The results of this article, it may be observed, are equally true of heterogeneous fluid, whatever be the law of variation of density in the successive strata.

153. *A mass of homogeneous liquid, the particles of which attract each other according to the Newtonian law, rotates uniformly, in a state of relative equilibrium, about an axis through its centre of gravity; required to determine a possible form of the surface.*

For the reason previously mentioned a direct solution of this problem cannot be obtained, but it can be shewn that an oblate spheroid is a possible form of equilibrium.

Let the equation to the spheroid be

$$\frac{z^2}{c^2} + \frac{x^2 + y^2}{c^2(1+\lambda^2)} = 1,$$

the axis of rotation being the axis of z.

Then the resultant attractions, towards the origin, on a particle at the point (x, y, z) will be represented by

$$X = \frac{2\pi\rho x}{\lambda^3}\{(1+\lambda^2)\tan^{-1}\lambda - \lambda\},$$

$$Y = \frac{2\pi\rho y}{\lambda^3}\{(1+\lambda^2)\tan^{-1}\lambda - \lambda\},$$

$$Z = \frac{4\pi\rho z}{\lambda^3}\{\lambda - \tan^{-1}\lambda\}(1+\lambda^2),$$

parallel, respectively, to the axes*.

We have then for the surfaces of equal pressure, putting ϵ for

$$\frac{\omega^2}{4\pi\rho},$$

$$\{2\epsilon\lambda^3 + \lambda - (1+\lambda^2)\tan^{-1}\lambda\}(xdx + ydy)$$
$$+ 2(\tan^{-1}\lambda - \lambda)(1+\lambda^2)\, zdz = 0.$$

But, from the equation to the spheroid,

$$xdx + ydy + (1+\lambda^2)\, zdz = 0,$$

and, as these equations must be identical,

$$2\epsilon\lambda^3 + \lambda - (1+\lambda^2)\tan^{-1}\lambda = 2(\tan^{-1}\lambda - \lambda);$$

an equation the roots of which determine the possible values of λ.

* These expressions will be found in Laplace's *Mécanique Céleste*, Poisson's *Mécanique*, Duhamel's *Mécanique*, and Todhunter's *Statics*. In the last named, the equation to the spheroid is $\frac{x^2+y^2}{a^2} + \frac{z^2}{a^2(1-e)} = 1$, but the expressions used in the text will result from the expressions there given by putting $1 - e^2 = \frac{1}{1+\lambda^2}$.

By the use of λ, irrational quantities are avoided.

It may be written

$$\frac{3\lambda + 2\epsilon\lambda^3}{3+\lambda^2} - \tan^{-1}\lambda = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (\alpha),$$

and the question is reduced to the discussion of the roots of this equation.

For this purpose consider the curve

$$y = \frac{3x + 2\epsilon x^3}{3+x^2} - \tan^{-1} x \;\ldots\ldots\ldots\ldots\ldots\; (\beta).$$

The abscissæ of the points where this curve cuts the axis will be the values of λ required.

It must be observed that, in the equation (α), $\tan^{-1}\lambda$ is the least positive angle whose tangent is λ; we have therefore only to consider one branch of the curve (β).

If the signs of x and y be changed, the equation is unaltered; the curve is therefore the same in the compartment $-x, -y$, as in $+x, +y$, and it is sufficient to examine the nature of the positive portion of the branch.

When $x = 0$, $y = 0$, and as x increases from zero, y begins by being positive, and when x increases indefinitely, has always positive values; hence the curve cuts the axis of x in an even number of points, exclusive of the origin.

Again, $$\frac{dy}{dx} = \frac{2x^2\{\epsilon x^4 + 2(5\epsilon - 1)x^2 + 9\epsilon\}}{(1+x^2)(3+x^2)^2},$$

$\frac{dy}{dx}$ is therefore zero at the origin (a point of inflection), and also at the points given by

$$\epsilon x^4 + 2(5\epsilon - 1)x^2 + 9\epsilon = 0 \;\ldots\ldots\ldots\ldots\ldots\; (\gamma).$$

If the values of x^2, obtained from this equation, be real, and positive, there will be a maximum and a minimum value of y; the former, corresponding to the smallest root, will evidently be positive, since y begins by being positive; if the latter, corresponding to the greatest root, be negative or zero, there will be two zero values of y or one only, and consequently two possible spheroidal forms of equilibrium, or one only.

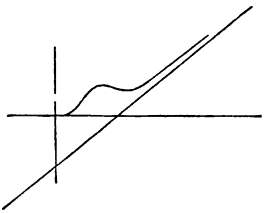
If the minimum value of y be positive, there will be no zero value of y; that is, the equilibrium of the fluid in the form of a spheroid is impossible.

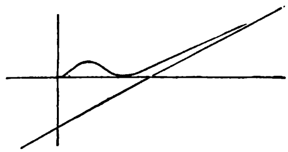
154. The preceding investigation may be illustrated by tracing the curve (β) for different values of ϵ.

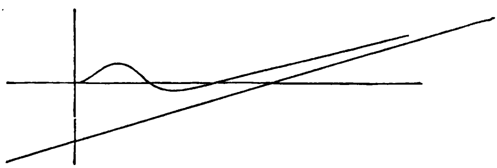
Putting $\tan^{-1} x = \frac{\pi}{2} - \tan^{-1}\frac{1}{x}$, and expanding, we obtain

$$y = 2\epsilon x - \frac{\pi}{2},$$

as the asymptote of the branch of the curve under consideration, and the appended figures exemplify the different cases above mentioned.







Numerical Calculation.

155. *To calculate the limiting value of ω for which the spheroidal form is possible.*

The equation (γ) may have positive roots if $5\epsilon < 1$; moreover the values of x^2 will be real, and positive, if

$$(1 - 5\epsilon)^2 > 9\epsilon^2, \text{ or } 1 - 5\epsilon > 3\epsilon;$$

$$\text{i.e. } \epsilon < \tfrac{1}{8}.$$

The superior limiting value of ϵ can however be obtained very approximately from the condition that, in the extreme case of possibility, the minimum value of y is zero.

We have then $y = 0$ and $\dfrac{dy}{dx} = 0$, simultaneously.

Hence, substituting in (β) the value of ϵ obtained from (γ), and putting $y = 0$, we have

$$\frac{x(7x^4 + 30x^2 + 27)}{(x^2+1)(x^2+9)(x^2+3)} - \tan^{-1} x = 0,$$

or

$$\frac{x(7x^2+9)}{(x^2+1)(x^2+9)} - \tan^{-1} x = 0 \quad \ldots\ldots\ldots\ldots (\delta).$$

An approximate value of the positive root of this equation will be a value of x, which, substituted in (γ), will give approximately the superior limit of the value of ϵ.

Since $\omega^2 = 4\pi\rho\epsilon$ this determines the greatest possible rate of rotation consistent with the existence of a spheroidal form.

When ω is less than the limiting value thus obtained, there will be two spheroids, either of which will be a possible form of the rotating fluid.

156. *Approximate determination of the positive root of the equation*

$$\frac{x(7x^2+9)}{(x^2+1)(x^2+9)} - \tan^{-1} x = 0.$$

Denoting the first member by $f(x)$, it will be found that

$$f'(x) = \frac{8x^4(3-x^2)}{(x^2+1)^2(x^2+9)^2}:$$

this is positive from $x = 0$ to $x = \sqrt{3}$, and is afterwards negative; $f(x)$ therefore increases until $x = \sqrt{3}$, and then diminishes; and, since $f(0) = 0$, $f(x)$ begins by being positive.

By the use of the formulæ

$$\tan^{-2} 2 = \frac{\pi}{4} + \tan^{-1} \tfrac{1}{3}$$

$$\tan^{-1} 3 = \frac{\pi}{4} + \tan^{-1} \tfrac{1}{2},$$

it will be found without much difficulty that the root lies between 2 and 3, but the application of Newton's method with the value 2 as an approximate one shews that a closer limit will be convenient.

If then 2·5 be substituted we obtain, by the aid of the formula

$$\tan^{-1}(2{\cdot}5) = \tan^{-1}(2) + \tan^{-1} \tfrac{1}{12},$$

$$f(2{\cdot}5) = {\cdot}0025 \text{ approximately.}$$

Let $$x = 2{\cdot}5 + y,$$

then, approximately, $$y = -\frac{f(2{\cdot}5)}{f'(2{\cdot}5)},$$

but $$f'(2{\cdot}5) = -{\cdot}085, \text{ nearly};$$

$$\therefore y = {\cdot}0293 \text{ and } x = 2{\cdot}5293.$$

The substitution of this value in (γ) will give

$$\epsilon = {\cdot}1123,$$

as the greatest possible value of ϵ or $\dfrac{\omega^2}{4\pi\rho}$.

Hence, when ω is such that $\epsilon < {\cdot}1123$, there are two spheroidal forms of equilibrium.

If ϵ is very small, one of the values of x (i.e. λ) will be very small and the other large, and therefore as ϵ decreases, the one spheroid becomes *very oblate* and approximates to a plane lamina, while the other approaches to the form of a sphere.

To find the small value of λ which satisfies the equation

$$\frac{3\lambda + 2\epsilon\lambda^3}{3 + \lambda^2} - \tan^{-1}\lambda = 0,$$

expand in ascending powers of λ, and we obtain

$$\lambda^2 = \frac{15\epsilon}{2} \text{ approximately.}$$

This gives a spheroid very slightly oblate, the ratio of its axes being $\sqrt{(1+\lambda^2)} : 1$, or very nearly $1 + \frac{15\epsilon}{4} : 1$.

The *large* value of λ is obtained by putting

$$\tan^{-1}\lambda = \frac{\pi}{2}\tan^{-1}\frac{1}{\lambda},$$

and expanding in powers of $\frac{1}{\lambda}$, a process which gives

$$\lambda = \frac{\pi}{4\epsilon} - \frac{8}{\pi} + \text{terms involving positive powers of } \epsilon,$$

as an approximation.

157. *Application to the case of a fluid, the density of which is equal to the Earth's mean density.*

If r be the Earth's radius and ρ the mean density of the Earth,

$\frac{4}{3}\pi\rho r$ is the attraction at the surface of a sphere of fluid of the same radius as the Earth, and of density ρ.

Suppose for a moment that the Earth is homogeneous, and spherical,

then $\frac{4}{3}\pi\rho r$ measures the force of gravity at the pole.

But, since $\epsilon = \frac{\omega^2}{4\pi\rho}$, and therefore $3\epsilon = \frac{\omega^2 r}{\frac{4}{3}\pi\rho r}$,

$3\epsilon : 1 ::$ difference of the measures of gravity at the pole and the equator : gravity at the pole (g).

Taking a second and a foot as the units of time and space, $g = 32$ approximately, $r = 4000 \times 1760 \times 3$, and it will be found that the time of rotation, $\frac{2\pi}{\omega}$, giving by the limiting value $\cdot 1123$ of ϵ, is a little more than $\frac{1}{10}$th of a day.

This then is the smallest time in which a homogeneous fluid mass, of density equal to the Earth's mean density, could rotate uniformly so as to be spheroidal in form.

158. The Earth, as is known by geodetic measurements, differs very slightly in its form from a sphere, and we can therefore apply our equations with great ease to the question of the homogeneity of the Earth, assuming it to have taken its present form when in a state of fluidity, or to be now a mass of fluid contained within a comparatively thin crust.

It has been found by observation, that for the Earth the ratio $\omega^2 r : g$ is about $1 : 289$, and we have therefore

$$3\epsilon = \tfrac{1}{289}.$$

But from Art. (156), $\lambda^2 = \dfrac{15\epsilon}{2} = \tfrac{5}{578}$,

and the ratio of the axes of the spheroid

$$= 1 + \frac{\lambda^2}{2} : 1 = 232 : 231, \text{ nearly.}$$

This result does not accord with the facts obtained by actual measurement, which give $301 : 300$ as an approximate value of the ratio.

The inference is that the Earth is not homogeneous.

159. The foregoing articles are taken chiefly from Laplace, *Mécanique Céleste*, Tome II.

It must be observed that the general problem of the form of a rotating fluid is not solved; all that is shewn being that, in certain cases, an oblate spheroid is a possible form of equilibrium.

If ω be such that $\epsilon > \cdot 1123$, it does not follow that equilibrium is impossible, but only that the spheroidal form cannot exist for that particular angular velocity.

If we put $-\lambda'^2$ for λ^2, taking λ'^2 as a positive quantity less than unity, the equation (γ) of Art. 153 becomes

$$\epsilon\lambda'^4 - 2(5\epsilon - 1)\lambda'^2 + 9\epsilon = 0,$$

or $$\epsilon(1-\lambda'^2)(9-\lambda'^2)+2\lambda'^2=0,$$
an equation which has no root less than unity.

From this it follows that a prolate spheroid is not a possible form of equilibrium*.

160. An important distinction has been pointed out by Poisson (Tome II. p. 547), between the surfaces of equal pressure in a fluid at rest under the action of extraneous forces, and in a fluid at rest, or revolving uniformly about a fixed axis, under the action of the mutually attractive forces of its particles.

Let ABC be the free surface, and DEF any surface of equal pressure; then, in the former case, the resultant force at any point of DEF is perpendicular to the surface at that point, and is unaffected by the existence of the fluid between ABC and DEF; this fluid could therefore be removed without affecting the equilibrium of the fluid mass bounded by DEF. In the latter case, the force at any point of DEF, although perpendicular to the surface at that point, is the resultant of the attractions of the mass of fluid contained by DEF, and of the mass contained between DEF and ABC; these two components of the resultant force are not necessarily perpendicular to the surface, and the fluid external to DEF cannot in general be removed without affecting the equilibrium of the remainder.

If, however, the fluid be homogeneous, and the particles attract each other according to the Newtonian law, so that the free surface may be spheroidal, the surfaces of equal pressure will be similar spheroids; and in this case, since the resultant attraction of an ellipsoidal shell on an internal particle is zero, the portion of fluid between ABC and DEF may be removed, provided the rate of rotation remain unaltered.

Moreover we have shewn, Art. (153), that for a given value of ω not exceeding a determined limit, there are two possible spheroidal forms: let ABC, the free surface, have

* *Méc. Céleste*, Tom. II. p. 59. The proof is also given in Pontécoulant's *Système du Monde*, Tom. II. p. 401.

one of these forms, and describe within the fluid mass a concentric spheroid, GHK, similar to the other spheroid; then the fluid between ABC and GHK may be removed without affecting the fluid mass GHK.

The action of the shell upon a particle at a point P of the surface GHK is not perpendicular to the surface at P, but this action, combined with the attraction of the mass GHK, and the hypothetical force measured by $\omega^2 r$, is perpendicular to the surface, at P, of the spheroid passing through P, which is concentric with, and similar to, the surface ABC.

161. If a fluid mass be set in motion, about an axis through its centre of gravity, with an angular velocity such as to make the value of ϵ greater than the limit obtained in Art. (155), it does not follow that the fluid cannot be in equilibrium in the form of a spheroid, for it may be conceived that the mass will expand laterally with reference to the axis, taking a more flattened shape, until its angular velocity is so far diminished as to render the spheroidal form possible.

If the mass consist of perfect fluid, its form will oscillate through the spheroid of equilibrium, but if, as is the case in all known fluids, friction be called into play by the relative displacement of the particles, the oscillations will gradually diminish and at length a position of equilibrium will be attained. Employing the principle that the 'Angular momentum' of the system, relative to the axis, will remain constant, we can determine the final angular velocity, and the form ultimately assumed*.

Considering the question generally, suppose the mass of fluid set in motion in any way, and then left to itself; the centre of gravity will be either at rest or moving uniformly in a straight line, and all we have to consider is the motion relative to the centre of gravity.

Draw through the centre of gravity the plane, in the direction of which the angular momentum is a maximum; then, however during the subsequent motion the fluid

* The angular momentum of a system, relative to an axis, is the sum of the moments of the momenta of the several particles of the system about the axis.

particles act on each other, this plane, which may be called the 'momental' plane, will remain fixed, and when the motion of the particles relative to each other has been destroyed by their mutual friction, the axis perpendicular to this plane will be the axis of rotation of the fluid mass in its state of relative equilibrium.

Let $2H$ be the given angular momentum of the system, and ω its ultimate angular velocity.

Taking c and $c\sqrt{(1+\lambda^2)}$ for the axes of the spheroid of equilibrium, and M for the mass, the expression for the angular momentum is $\frac{2}{5}Mc^2(1+\lambda^2)\omega$;

$$\therefore \tfrac{1}{5}Mc^2(1+\lambda^2)\omega = H;$$

we have also $$\tfrac{4}{3}\pi\rho c^3(1+\lambda^2) = M,$$

and from these two equations, combined with the equation,

$$\frac{3\lambda+2\epsilon\lambda^3}{3+\lambda^2} - \tan^{-1}\lambda = 0 \;\ldots\; \text{Art. (153)},$$

the values of c, ω, and λ can be determined.

From the first two we obtain

$$\epsilon = \frac{\omega^2}{4\pi\rho} = \frac{25H^2(\frac{4}{3}\pi\rho)^{\frac{1}{3}}}{3M^{\frac{10}{3}}}(1+\lambda^2)^{-\frac{2}{3}},$$

$$= p\,(1+\lambda^2)^{-\frac{2}{3}}, \text{ supposed};$$

$$\therefore \frac{3\lambda+2p\lambda^3(1+\lambda^2)^{-\frac{2}{3}}}{3+\lambda^2} - \tan^{-1}\lambda = 0,$$

is the equation which determines λ.

The left-hand member of this equation is positive when λ is very small, and negative when λ is indefinitely large, and the equation has therefore a positive root; consequently, the fluid mass will at length attain a spheroidal form of equilibrium.

It can be shewn moreover that the equation has only one positive root, and therefore there is one spheroidal form, and one only, towards which the oscillating fluid mass continually approximates.

This discussion is taken from the *Mécanique Céleste*, Tom. II. p. 71, and from Pontécoulant's *Système du Monde*, Tom. II. p. 409.

162. It was discovered by Jacobi that an ellipsoid with three unequal axes is a possible form of relative equilibrium for a mass of rotating liquid.

The following proof of Jacobi's theorem is taken from a paper by Liouville in the *Journal de l'École Polytechnique*, Tome XIV.

Taking the axis of rotation for the axis of z, suppose, if possible, that the surface of the liquid is of the form given by the equation

$$\frac{x^2}{1+\lambda^2}+\frac{y^2}{1+\lambda'^2}+z^2=c^2 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Then, if M be the mass of the liquid, the resultant attractions on a particle at the point (x, y, z) of the surface are respectively Ax, By, and Cz*,

where
$$A=\frac{3M}{c^3}\int_0^1\frac{u^2du}{(1+\lambda^2u^2)H},$$

$$B=\frac{3M}{c^3}\int_0^1\frac{u^2du}{(1+\lambda'^2u^2)H},$$

$$C=\frac{3M}{c^3}\int_0^1\frac{u^2du}{H},$$

H representing the expression

$$\sqrt{(1+\lambda^2u^2)(1+\lambda'^2u^2)}.$$

The differential equation of the free surface is

$$(Ax-\omega^2x)\,dx+(By-\omega^2y)\,dy+Czdz=0,$$

and therefore, if the free surface be the ellipsoid (1),

$$(A-\omega^2)(1+\lambda^2)=(B-\omega^2)(1+\lambda'^2)=C \quad \ldots\ldots (2).$$

Eliminating ω^2, we obtain

$$(1+\lambda^2)(1+\lambda'^2)(A-B)=C(\lambda'^2-\lambda^2),$$

* See the *Mécanique Céleste*, Tome II., or Duhamel's *Cours de Mécanique*.

and, substituting for A, B, and C, this reduces to

$$(1+\lambda^2)(1+\lambda'^2)\int_0^1 \frac{(\lambda'^2-\lambda^2)\,u^4du}{H^3}=(\lambda'^2-\lambda^2)\int_0^1\frac{u^2du}{H}.$$

Rejecting the solution $\lambda'=\lambda$, which leads to the case of an oblate spheroid, and transposing, we obtain

$$\int_0^1 \frac{u^2(1-u^2)(1-\lambda^2\lambda'^2u^2)\,du}{H^3}=0,$$

an equation which, if λ be assigned, determines λ'.

Assigning a positive value to λ^2, the left-hand member of the equation is positive if $\lambda'=0$, and is negative if $\lambda'=\infty$; hence there is a positive value of λ'^2 which will satisfy the equation.

Moreover, from the equations (2),

$$\omega^2 = A-\frac{C}{1+\lambda^2}$$

$$=\frac{3M}{c^3}\int_0^1 \frac{\lambda^2(1-u^2)\,u^2du}{(1+\lambda^2)(1+\lambda^2u^2)\,H},$$

and ω^2 is therefore a positive quantity.

Hence it is completely established that an ellipsoid with three unequal axes, the smallest of which coincides with the axis of rotation, is a possible form of the free surface.

163. The resultant action of gravity at the surface is the resultant of the forces $(A-\omega^2)\,x$, $(B-\omega^2)\,y$, and Cz, and is therefore inversely proportional to the perpendicular from the centre on the tangent plane.

Also, bearing in mind that the attractions of a solid on an internal particle are Ax, By, and Cz, and utilizing Leibnitz's theorem, it is easily shewn that the resultant stress across any central plane section is perpendicular to that plane, and proportional to its area.

164. It was pointed out by Mr Todhunter, and demonstrated in the following manner, that the relative equilibrium of the rotating ellipsoid cannot subsist when the axis of rotation does not coincide with a principal axis.

Referred to the principal axis, let l, m, n, be the direction cosines of the axis of rotation, M any point (x, y, z) of the mass, and N the foot of the perpendicular from M upon the axis.

Then $$ON = lx + my + nz,$$
and, if $ON = v$, the co-ordinates of N are lv, mv, nv.

The acceleration $\omega^2 MN$, when resolved parallel to the axes, gives rise to the components

$$\omega^2 (x - lv),\ \omega^2 (y - mv),\ \omega^2 (z - nv);$$

therefore the differential equation of the free surface is

$$\{\omega^2(x - lv) - Ax\}\, dx + \{\omega^2(y - mv) - By\}\, dy + \{\omega^2(z - nv) - Cz\}\, dz = 0;$$

hence the form of the free surface is given by the equation,

$$\omega^2(x^2 + y^2 + z^2) - \omega^2(lx + my + nz)^2 - Ax^2 - By^2 - Cz^2 = \text{constant},$$

and this cannot represent an ellipsoid referred to its principal axes, unless two of the quantities l, m, n, vanish.

Mr Greenhill remarks that a particle of the liquid at the end of the axis of rotation will be at rest under the action of the attraction of the liquid alone, since the centrifugal force at that point vanishes.

Hence the attraction on the particle must be normal to the surface, which is only the case at the end of an axis.

MISCELLANEOUS EXAMPLES.

1. A cone, with its axis inclined at an angle θ to the vertical, contains some water; it is turned till its axis is vertical. Shew that the whole pressure is altered in the ratio $\cos\theta : 1$.

2. If mercury is gradually poured into a vessel of any form containing water, prove that the centre of gravity of the mercury and water will be in its lowest position when its height above the common surface bears to the depth of water the ratio of the density of water to that of mercury..

3. If fluid of which the density at depth z is $\rho_0(1+\lambda z)$ fill a hemispherical bowl, radius a, prove that the whole pressure is the same as if the fluid were uniform and of density equal to that at a depth $\frac{1}{3}a$.

4. The interior of a port wine glass is in the form of a paraboloid of revolution whose height is equal to its latus rectum. If the glass be filled with homogeneous liquid, shew that the whole pressure on the glass is to the weight of the liquid as

$$25\sqrt{5}-11 : 30.$$

5. Two buckets containing water, the mass of each bucket with the contained water being M, balance each other over a smooth pulley. Two pieces of wood of masses m, m', and specific gravities σ, σ' are then tied to the bottoms of the buckets so as to be wholly immersed, prove that the tension of the string attached to the mass m is

$$\frac{2m(M+m')g}{2M+m+m'}\left(\frac{1}{\sigma}-1\right).$$

6. A quantity of elastic fluid whose particles attract one another according to the law of nature fills a sphere in whose centre resides a central force $\frac{\mu}{\rho}$. The radius of the sphere is c and the mass of fluid $(2\kappa - \mu)\,c$, where $\kappa = \frac{p}{\rho}$. Shew that the conditions of equilibrium are satisfied if $\rho \propto$ inversely as r^2.

7. A sphere (radius c) is just filled with water, and rotates about a vertical axis with angular velocity ω, such that $3c\omega^2 = 2g$; prove that the pressure in the surface of equal pressure which cuts the sphere at right angles is $3g\rho c \div 4$, ρ being the density of water.

8. A small quantity of fluid is spread over the surface of a material prolate spheroid. Shew that the free surface of the fluid is also a spheroid, and that the depth of the fluid at the equator is to the depth at the pole as the major axis of the spheroid to the minor.

9. A spherical shell, whose interior radius is a, is filled with liquid of uniform density ρ, and revolves with uniform angular velocity ω about the vertical diameter of the shell; shew that, if the total normal pressure on the upper half of the shell be to that on the lower half as $m : n$, the pressure at the highest point of the liquid is

$$\rho \left\{ \frac{3m - n}{n - m} \frac{ga}{2} - \frac{\omega^2 a^2}{3} \right\}.$$

10. A hollow sphere, filled with equal quantities of two liquids which do not mix, revolves uniformly about its vertical diameter, and the liquid particles are relatively at rest. Find the angular velocity when the lighter liquid just touches the lowest point in the surface of the sphere.

11. A hollow cylinder is filled with water and made to revolve about a vertical axis attached to the centre of its upper plane face with a velocity sufficient to retain it at the same inclination to the axis. Find at what point of the surface a hole might be bored without loss of fluid.

12. A mass of liquid is contained between three co-ordinate planes, each of which attracts with a force varying as the distance, and the absolute forces of attraction μ, μ', μ'' are in harmonic progression. Half an ellipsoid is fixed with its plane face against one of the co-ordinate planes, and its surface touching the other planes, its axis being parallel to the co-ordinate axes and proportional to

$$\frac{1}{\sqrt{\mu}},\ \frac{1}{\sqrt{\mu'}},\ \frac{1}{\sqrt{\mu''}}.$$

If there be not sufficient fluid quite to cover the ellipsoid, the uncovered part will be bounded by a circle.

13. A mass of liquid is subject to the mutual gravitation of its particles, and to a repulsive force tending from a plane through its centre of gravity and varying as the perpendicular distance from that plane; shew that the conditions of equilibrium will be satisfied if the surface be a prolate spheroid of a certain ellipticity, provided the repulsive force be not too great.

14. A right cylindrical vessel on a plane base contains a certain quantity of gas, which is confined within it by a disc exactly similar and parallel to the base; shew that the pressure on the curved surface of the cylinder is independent of the position of the disc.

15. A cup floats upright in oil, and is ballasted with water: find its form and sketch it, when the difference of level of the two liquid surfaces is the same for all degrees of immersion.

16. If a liquid be inclosed in a vessel of any form and be allowed to run into another vessel of different form, and if p be the pressure at x, y, z, in either of the vessels referred to rectangular co-ordinates independent of them, the difference between the two values of $\iiint p\,dx\,dy\,dz$ differs from the work done by the liquid in running from the upper to the lower vessel by the work required to bring the surface of the fluid in the lower vessel to the same horizontal plane with the original surface in the upper.

17. A vessel in the form of a hemisphere with a plane lid is held with the lid vertical. Prove that the resultant pressure on the curved surface acts along a radius inclined at an angle $\tan^{-1}\frac{3}{2}$ to the vertical. If now the hemisphere and fluid be made to rotate about the vertical diameter with uniform angular velocity ω, shew that the resultant pressure due to the rotation only on the curved surface is three times as great as the resultant pressure due to the rotation only on the plane surface, and that the whole resultant pressure on the curved surface now acts along a radius inclined to the vertical at an angle

$$\tan^{-1}\tfrac{3}{8}\,.\,\frac{8g+3\omega^2 r}{2g}.$$

18. A vessel in the shape of a paraboloid of revolution contains some fluid which is rotating about the vertical axis of the paraboloid. Find the angular velocity when the fluid begins to spill, and shew that, if this is $\sqrt{\frac{g}{l}}$, the vessel must have been half full of fluid.

If the paraboloid be not of revolution but of the form $z=\frac{x^2}{l}+\frac{y^2}{l'}$, the axis of z being vertical, and if z_1, z_2 be the greatest and least heights of the curve in which the surface of the fluid meets the vessel, prove that

$$\frac{1}{z_2}-\frac{1}{z_1}=\frac{\omega^2}{2gc}(l\sim l'),$$

where c is the distance between the vertices of the two paraboloids.

19. A hollow vessel in the form of an anchor ring, just filled with water, spins uniformly round the vertical axis of generation, the whole moving as a solid ring. Find the whole pressure on the internal surface.

20. A cylindrical diving-bell is suspended with its axis vertical at a depth such that the water rises half-way up the bell: find the least distance of the centre of gravity of the

bell from the centre of its upper surface, consistent with the condition that the equilibrium may be stable with reference to an angular displacement of the axis.

21. A cylinder makes vertical oscillations in a liquid contained in another cylinder, the radius of which is n times that of the former; shew that the depth of the axis immersed when in a position of rest is $gt^2n^2 \div \pi^2 (n-1)$ where t is the time of an oscillation.

22. A vessel in the form of a paraboloid with its axis vertical, contains a quantity of liquid equal in volume to that of a segment of a paraboloid, of the same latus rectum, floating in it: if this be raised till its vertex is just in the surface, and if it then sink to a depth equal to $\frac{3}{4}$ of its axis before returning, shew that the density of the liquid : that of the paraboloid :: 48 : 7.

23. A closed cylindrical vessel one foot in height is half full of water, the other half being occupied by atmospheric air; if two small apertures be made, one at the base of the cylinder and the other five inches above it, shew that the density of the air in the vessel will decrease until it is $\left(1 - \frac{1}{12h}\right)$ times its original value approximately, and then increases again, h being the height of a water-barometer in feet.

24. Incompressible fluid is at rest under the action of forces

$$-\frac{\mu x}{a^2}, -\frac{\mu y}{b^2}, -\frac{\mu z}{c^2},$$

respectively parallel to the axes, and a particle, the density of which is less than that of the fluid, is placed anywhere in the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = m;$$

prove that, neglecting the resistance, the velocity of the particle when crossing the surface defined by the quantity m' varies as

$$\sqrt{m' - m}.$$

25. If the particles of a mass of liquid rotating uniformly about a fixed axis, attract one another according to such a law that the surfaces of equal pressure are similar coaxial oblate spheroids; prove that the resultant attraction of a spheroid, the particles of which attract according to the same law, is the resultant of two forces perpendicular to the equator and the axis of revolution respectively, and varying as the distance of the attracted point from them.

26. An elastic spherical envelope is in equilibrium when it contains air at twice the atmospheric density, and its radius is twice the natural size; if the barometer fall $\frac{1}{n}$th of an inch, find the time of a small oscillation in the magnitude of the envelope.

27. A right cone rests in a vessel containing equal depths of two given fluids, with its vertex fastened to the bottom and its axis vertical. Find the condition for stable equilibrium.

28. A straight uniform rod consisting of matter attracting as $(\text{dist.})^{-1}$ is surrounded by fluid at rest subject to its attraction only; shew that the differential equation to the meridian sections of the surfaces of equal pressure can be put in the form

$$\frac{dy}{dx} \cdot \psi + \log \frac{r}{r'} = 0,$$

r, r' being the distances of the point xy from the ends of the rod, and ψ the angle subtended by the rod at that point.

29. If $\alpha, \beta, \gamma, \delta$ be the depths of the corners of a quadrilateral area which is wholly immersed in liquid, and h the depth of its centre of gravity, the depth of its centre of pressure is

$$\tfrac{1}{2}(\alpha + \beta + \gamma + \delta) - \frac{1}{6h}(\beta\gamma + \gamma\alpha + \alpha\beta + \alpha\delta + \beta\delta + \gamma\delta).$$

30. A rigid spherical envelope of radius a is filled with elastic fluid of mass M which is acted on by a repulsive

force $=\mu$ (dist.)^2 from a point in the surface of the envelope: shew that the total normal pressure on the envelope is

$$4\mu a^2 M \frac{\int_0^1 x\epsilon^{\frac{8\mu a^3}{3\kappa}x^3}\,dx}{\int_0^1 \left(\epsilon^{\frac{8\mu a^3}{3\kappa}x^3} - 1\right)dx}.$$

31. A portion of a paraboloid, latus rectum $4a$, is cut off by a plane perpendicular to the axis at a distance $3a$ from the vertex; if the vertex of the paraboloid be fixed at a depth $\frac{\sqrt{3}}{2}a$ beneath the surface of a liquid, shew that it will rest with the focus in the surface if the ratio of the density of the liquid to that of the solid be 729 : 232.

32. An embankment of triangular section ABC supports the pressure of water on the side BC: find the condition of its not being overturned about the angle A when the water reaches to B, the vertex of the triangle: and shew that, when the area of the triangle is reduced to the minimum consistent with stability for a given depth of water,

$$\tan C = \frac{\sqrt{s^2+2s+9}}{3-s},$$

$$\tan A = \frac{\sqrt{s^2+2s+9}}{s-1},$$

where s is the specific gravity of the embankment.

33. A mass (M) of fluid, in which the density at any point is the sum of a given constant quantity and a quantity bearing a given constant ratio to the pressure at that point, revolves about a fixed axis with a given constant angular velocity, and is attracted to a point in that axis by a given force which varies as the distance: find the form of the free surface; and shew that its least semi-diameter (b) is determined by the equation,

$$M = m\int_0^b e^{\frac{b^2-x^2}{c^2}} x^2 dx,$$

when m and c are given constants.

34. A centre of force, repelling inversely as the square of the distance, lies below the surface of a homogeneous inelastic fluid, which is also acted on by gravity and is at rest: the intensity of the force, at a point in the surface of the fluid vertically above its centre, is equal to that of gravity: prove that the external surface of the fluid has a horizontal asymptotic plane, and that the centre of force is environed by an internal cavity, the summit of which is at the external surface of the fluid.

Find the volume of the cavity in terms of its length.

35. A right prism on a square base has another prism, also on a square base, attached to it, so that their axes are coincident and sides parallel, and the whole floats on a fluid with their common plane in the plane of floatation. If the sides of the bases of the two prisms are in the ratio 2 : 1, find their limiting heights in order that the equilibrium may be stable.

36. A heavy cube is moveable about an axis, which passes through, and bisects, the opposite sides of one face; this axis being fixed horizontally within an empty vessel, so that the cube is suspended in the position of equilibrium, find the depth to which fluid must be poured in, so as to render the equilibrium unstable, and the greatest ratio of the densities of the cube and fluid, that this may be possible.

Supposing the cube half immersed and the equilibrium stable, find the time of a small oscillation.

37. A cylinder whose axis is vertical is floating in a fluid in which the density at any point varies as the n^{th} power of the depth; the cylinder is depressed till its upper end just coincides with the surface of the fluid, and on being let go it rises just out of the fluid; shew that, when the cylinder was floating, the depth immersed was to the height of the cylinder as 1 to $(n+2)^{\frac{1}{n+1}}$.

38. A spherical homogeneous solid earth, supposed to be fixed, is surrounded by a shallow sea, which is attracted by a distant fixed body; prove that, neglecting the attraction

of water on itself, the surface of the sea will remain spherical, but that its centre will deviate from the centre of the earth by a distance amounting to the same fraction of its radius that the attraction of the disturbing body is of the attraction of the earth on an element of the liquid.

39. If the earth be supposed spherical and covered with an ocean of small depth, and if the attraction of the particles of water on each other be omitted, the ellipticity of the ocean spheroid will be given by the equation,

$$2\epsilon = \frac{\text{centrifugal force at the equator}}{\text{force of gravity at the earth's surface}}.$$

40. An isosceles triangular lamina, of which the sides AB, AC are equal, floats with the angular point downwards in a liquid of which the density varies as the depth: if AD be perpendicular to BC, prove that if the lamina can float with the line AD inclined at an angle θ to the vertical, θ is given by the equation,

$$81\sigma \sin^3 \theta = 64\rho \cos^8 \alpha (\sin^2 \theta - \sin^2 \alpha)^2,$$

where 2α is the angle BAC, σ is the density of the lamina, and ρ is the density of the liquid at a depth equal to AB or AC.

41. A solid of revolution floats with its axis vertical, and is sunk to different depths by placing weights at a fixed point of its axis. Find the form when the equilibrium is always neutral.

42. If a body float at rest, shew that for any displacement, consistent with the condition that the weight of the fluid displaced be equal to that of the float, the difference of the distances of the centres of gravity of the float and of the fluid displaced below the surface of the fluid will, in general, be a maximum or minimum according as the equilibrium is unstable or stable.

Moreover if Z be this difference, and the body be symmetrical with respect to a vertical plane, perpendicular to the line about which the displacement aforesaid is made, and θ

be the inclination of any fixed line in the body and in that plane to the vertical, the time of a small oscillation will be that of a simple pendulum of which the length is $\dfrac{k^2}{\dfrac{d^2Z}{d\theta^2}}$, where k is the radius of gyration about a line through the centre of gravity parallel to the axis of displacement.

Mention any conditions which limit the generality of these theorems.

43. An ellipsoid floats with the least axis $(2c)$ vertical in a fluid of twice its density, and makes small oscillations in a vertical plane about a point in the major axis $(2a)$ which is fixed. Shew that the period is

$$2\pi\sqrt{\frac{8}{15}\frac{c}{g}\frac{5\kappa^2+a^2+c^2}{\kappa^2+ac}},$$

where κ is the central distance of the fixed point.

44. A pneumatic railway carriage can move freely without friction in a tunnel which it exactly fits. It is placed at rest at one end, and an engine begins to exhaust the air at the other, pumping out equal volumes in equal times.

Shew that at time t the distance of the carriage from the end to which it is travelling is determined by an equation of the form

$$x\frac{d^2x}{dt^2}+b\frac{dx}{dt}+n(x+bt)=na.$$

45. A solid of revolution possesses this property. A portion being cut off by a plane perpendicular to its axis and immersed vertex downwards in a liquid and then displaced through a small angle, the moment tending to restore equilibrium is independent of the amount cut off. Shew that, if $y=f(x)$ be the generating curve, to determine f we have

$$[f(x)]^2=\rho\,[1+\{f'(x)\}^2+f(x)f''(x)]\,[f\{x+f(x)f'(x)\}]^2,$$

ρ being the density of the solid compared with the fluid.

46. If a given quantity of homogeneous matter be formed into a paraboloid of revolution and allowed to float in water with the vertex downwards, the square of the distance of the centre of gravity from the plane of floatation will be inversely proportional to the latus rectum.

47. A body, floating in a fluid, is turned through a small angle θ, round a principal axis at the centre of gravity of the plane of floatation; shew that the work done to produce the displacement is $\frac{1}{2} g\rho\theta^2 (Ak^2 - V.HG)$.

From a solid hemisphere, of radius r, a portion in the shape of a right cylinder, of height h, coaxial with the hemisphere and having the centre of its base at the centre of the hemisphere, is removed. Into this portion is fitted a thin tube which exactly fits it. The solid is placed with its vertex downwards in a fluid, and a fluid, of density ρ, is poured into the tube. Find how much must be poured in, in order that the equilibrium may be neutral; and if the tube be filled to a height $2h$, shew that

$$\frac{\rho}{s} = \frac{r^4 - 2b^2h^2}{b^4},$$

s being the density of the solid.

48. A solid body is floating in a liquid of variable density and its position is slightly changed so that the mass of liquid displaced remains unaltered. If $f(z)$ be the density at a depth z, and (x, y, z) the coordinates of any point in the immersed surface of the body, referred to the surface as the plane xy, prove that the point in the plane of floatation about which the body turns is the centre of gravity of that plane treated as a lamina, the density of which at the point (x, y) is $f(z)$.

49. A cup whose outside surface is a paraboloid of revolution of latus rectum l, and whose thickness measured horizontally is the same at every point and very small compared with l, has a circular rim at a height h above the vertex, and rests on the highest point of a sphere of radius r. If water be now poured in until its surface cuts the axis of the cup at a distance $\frac{3}{20}h$ from the vertex, and if the

weight of water be four times that of the cup, shew that the equilibrium will be stable, if

$$\frac{h}{l} < \frac{r-2l}{2r+l}.$$

50. An isosceles triangular lamina ACB is at rest with its plane vertical, and its vertex C fixed at a depth c below the surface of a liquid, the density of which varies as the depth. If the density of the lamina be the same as that of the liquid at the depth d, and if θ be the angle which the altitude h of the triangle makes with the vertical, prove that

$$8dh^3 \cos^2 \overline{\theta + \alpha} \, . \cos^2 \overline{\theta - \alpha} = 3c^4 \cos^2 \alpha \, . \cos \theta,$$

the angle ACB being 2α.

51. If a solid of revolution be immersed in a heavy homogeneous fluid with its axis vertical, prove that, when the total normal pressure on the surface is a minimum, its form must be such that the numerical value of the diameter of curvature of the meridian at any point is a harmonic mean between the segments of the normal to the surface at that point intercepted between the point and the surface of the fluid and between the point and the axis, respectively.

52. A hollow cylinder of height $2h$ and radius c with both ends closed contains water, and is placed with the centre of its base in contact with the highest point of a rough sphere of radius r; the weight of the water is equal to that of the cylinder, shew that the equilibrium will be stable if the water occupy a length of the cylinder which lies between the roots of

$$2x^2 - 4(2r - h)x + c^2 = 0.$$

53. A parabolic lamina, bounded by a double ordinate perpendicular to the axis, floats in a liquid with its focus in the surface and its axis inclined at an angle $\tan^{-1}\frac{\sqrt{7}}{2}$ to the vertical; prove that the density of the liquid is to that of the lamina as 216 : 121, and that the length of the bounding ordinate is three times the latus rectum.

54. A weightless shell in the form of a paraboloid of revolution rests in a similar shell, the parameter of which is double that of the former, and contains fluid whose density varies as $(\text{depth})^n$. Find the depth of the fluid in order that the equilibrium may be neutral.

55. A conical vessel of height h, vertex downwards, is filled with liquid the density of which is λx, x being the depth. This is poured into another vessel in the form of a surface of revolution, and it is found that the new density is μx^2. Prove that the form of the vessel is given by the equation,

$$y^2 + z^2 = \frac{2\mu}{\lambda} x \left(h - \frac{\mu}{\lambda} x^2\right)^2 \tan^2 \alpha.$$

56. An indefinitely small piece of ice, the shape of which may be taken to be that of a right circular cylinder, is floating with its axis vertical in water. The part immersed receives deposits of ice in such a manner as to continue cylindrical, the radius and axis receiving equal increments in equal times. Find the ultimate shape of the part not immersed.

If the specific gravity of ice be ·96, prove that the surface is formed by the revolution of $y^2 (9x - y)^{25} = a^{27}$.

57. A solid in the form of a paraboloid of revolution floats with its axis vertical; if the centre of gravity coincide with the metacentre, prove that the equilibrium is stable.

58. If when the barometer stands at 30 inches, the specific gravity of mercury being 13·596 referred to water, of which a cubic inch weighs 252·77 grains, a cubic yard of atmospheric air is compressed into a vessel containing a cubic foot, find approximately the numerical measure of the energy stored up therein.

59. The expansions of water and glass are given by the formulæ

$$V_t = V_4 \{1 + a(t-4)^2\}, \text{ and } V_t = V_0 (1 + 5at),$$

where t is the temperature centigrade. If a water thermo-

meter be constructed and graduated in the same way as the common mercurial thermometer, prove that except at the freezing and boiling points it will always give too low a reading; that that reading will be negative from 0^0 to a little over 13^0; and that the error will be a maximum when $5at^2+2t=100$.

60. A quantity of air, whose density is ρ and pressure p, is enclosed in a spherical vessel. Shew that if a centre of force μD^n be placed at the centre of the sphere the density at a distance r from the centre will be

$$\frac{n+1}{3}\frac{a^3}{\Gamma\left(\dfrac{3}{n+1}\right)}\left\{\frac{\mu\rho^{\frac{n+4}{3}}}{p(n+1)}\right\}^{\frac{3}{n+1}} e^{-\frac{\mu\rho}{p(n+1)}r^{n+1}},$$

the intensity of the force being supposed so great that the density of the air in contact with the vessel may be neglected.

61. The pressure and density of the atmosphere at the earth's surface being p_0, ρ_0 and the temperature at higher points varying inversely as the nth power of the distance from the centre of the earth; prove that the pressure at a distance r from the earth's centre is

$$p_0 e^{-\frac{g\rho^0}{(n-1)p_0a^{n-2}}(r^{n-1}-a^{n-1})}.$$

where a is the radius of the earth.

If $n=1$, shew that a spherical balloon of material equally extensible in all directions will have its volume greatest when r is given by the equation

$$p_0(m-1)\frac{a^m}{r^m}=\frac{2\lambda}{c}\left\{1-m^{\frac{1}{3}}\frac{a^{\frac{m-1}{3}}}{r^{\frac{m-1}{3}}}\right\},$$

when $m=\dfrac{g\rho_0a}{p_0}$, λ is the modulus of elasticity, and c is the unstretched radius of the balloon, it being just filled and unstretched when it leaves the ground.

62. A balloon is at a certain moment at a height h, descending with velocity V, and moving horizontally with a velocity V' equal to the velocity of the wind at that height. If the velocity of the wind be proportional to the height, and if with a view to descending at a particular spot, the escape of the gas be regulated so as to keep the velocity of descent constant, prove that a miscalculation dh in the initial height will produce in the point reached an error,

$$\frac{V'dh}{c^2V}\{1+\tfrac{1}{2}c^2-e^{-c}(1+c)\}, \text{ where } c=\frac{gh}{V^2}.$$

63. Prove that the work done during the $(n+1)^{\text{th}}$ stroke of a Smeaton's air pump, supposing the expansion to be isothermal, is equal to

$$\Pi\left(\frac{A}{A+B}\right)^n\left\{(nB+A)\log\left(1+\frac{B}{A}\right)+B\right\},$$

64. The condensation being isothermal, find the work done during the n^{th} stroke of a condenser.

65. A solid is composed of two cubes, symmetrically joined together, but of different material and size. It floats with the common plane in the surface of a fluid. Find the condition of stability.

66. A small spherical cavity (radius $=R$) in an attracting mass is filled with a homogeneous incompressible fluid, and the attraction at the centre of the sphere is evanescent: prove that the fluid pressure at the centre cannot be less than $-\frac{1}{2}\rho cR^2$, and the total pressure on the surface of the cavity not less than $-(c+\frac{4}{3}\pi\rho)\,2\pi\rho R^4$, where ρ is the density of the fluid, and, U denoting the potential of the attracting mass, c is the least algebraical value of $\frac{d^2U}{ds^2}$ at the centre for an element ds drawn in any direction from the centre.

67. A soap-bubble of uniform thickness is filled with a gas of such density that the weight of the whole is that of the air displaced; find the form of the bubble, which is supposed to differ but little from a sphere.

68. A slender fluid ring revolves uniformly round a centre of force situated at its centre, the force varying inversely as the square of the distance; find approximately the form of a section of the ring.

69. A vessel of given capacity, in the form of a surface of revolution with two circular ends, is just filled with inelastic fluid which revolves about the axis of the vessel, and is supposed to be free from the action of gravity: investigate the form of the vessel that the whole pressure which the fluid exerts upon it may be the least possible, the magnitudes of the circular ends being given.

Shew that, for a certain relation between the radii of the circular ends, the generating curve of the surface is the common catenary.

70. Employ the principle of energy to find the equation of the Lintearia.

71. Prove that the equation,

$$\epsilon^z(\epsilon^x + \epsilon^{-x}) = \epsilon^y + \epsilon^{-y},$$

represent a possible form of a liquid film, the pressure on both sides being the same. (*Catalan.*)

72. A cylindrical vessel, the cross section of which is formed of two cycloidal arcs with the ends fitting together, has an excess of air pressure inside; investigate the stresses along any generating line.

73. If the particles of a spherical soap-bubble, of radius r and tension t, repel each other according to the law of the inverse square of the distance, and if V be the potential, prove that $V^2 = 16\pi rt$.

74. Into a spherical brass shell, of radius a, water is forced until the radius of the shell is found to expand to r. Having given that the coefficient of elasticity of the shell for stretching is μ, and that the compressibility of water is λ; shew that the quantity of water in the shell is

$$\tfrac{4}{3}\pi\rho \frac{ar^4}{ar - 2\lambda\mu(r-a)}$$

where ρ is the density of uncompressed water.

In the previous question you are given $a=4$, $r=5$ centims.: and the following data:—

The compression of water for 1 atmos. (1 megadyne per sq. cm.) $=10^{-5}\times\cdot 5$; thickness of shell $=\cdot 5$ mm., and a brass wire of 1 sq. mm. section requires a force of 9000 megadynes to double its length, if its elasticity remain constant. Determine the measures in C.G.S. units of the quantities involved, and thence shew that the mass of water in the sphere $=535$ grams approximately.

75. A soap-bubble of uniform thickness is filled with a gas of such density that the weight of the whole is equal to that of the air displaced; find the form of the bubble, which is supposed to differ but little from a sphere.

76. A soap-bubble extends from fixed boundaries, so as with them to form a closed space whose volume is v_0 and contains a gas at pressure p_0 and absolute temperature θ_0. The temperature of the gas is gradually raised. If A be the area of the film when the temperature is θ, and pressure p, shew that

$$t\theta_0\frac{dA}{d\theta}=p_0v_0\left(1-\frac{\theta}{p}\frac{dp}{d\theta}\right)$$

where t is the surface-tension supposed constant, and the external pressure is neglected.

Find the relation between p and θ in the case of a spherical soap-bubble, and in this case integrate the above.

77. Two equal circular discs of radius a are placed with their planes perpendicular to the line which joins their centres, and their edges are connected by a soap film which encloses a mass of air that would be just sufficient in the same atmosphere to fill a spherical soap-bubble of radius c. If the film be cylindrical when the distance between the discs is b, prove that in order that it may become spherical the distance between the discs must be lessened to $2z$ where

$$z(3a^2+2z^2)\left\{8c^2-3ab+\frac{6a^2b-8c^3}{\sqrt{a^2+z^2}}\right\}=6abc^2(2a-c).$$

78. A hemispherical bubble is floating on water. Assuming that its radius is such that the ratio of the difference of the internal and external pressures to the external pressure is a small quantity whose square may be neglected, find the form of the water surface inside the bubble, and shew that its greatest depression below the external water surface is

$$\frac{2a^2}{r}\left\{1-\frac{2\pi}{\int_{-\pi}^{\pi}\epsilon^{\frac{r}{a}\sin\phi}\,d\phi}\right\},$$

where r is the radius of the bubble and a^2 the ratio of the surface energy for air and water per unit area to the weight of unit volume of water. (*Mr Burnside.*)

79. Giffard's freezing machine consists of two cylinders, the pistons of which work on to two cranks on the same shaft, driven by an external source of power, and of a large air reservoir which is always maintained at the temperature of the external air. In the first cylinder air is compressed till its pressure is the same as in the reservoir, when valves open and the air passes, as the stroke is completed, into the reservoir. The second and smaller cylinder acts as an engine receiving compressed air from the reservoir for such a portion of the stroke that being expanded for the remainder of the stroke it is discharged at atmospheric pressure, but at a lower temperature. If V_1 and V_2 be the volumes of the cylinders, and if the compression and expansion be supposed adiabatic, prove that the work done during each stroke in the first cylinder is $\Pi V_1 \dfrac{\gamma}{\gamma-1} \cdot \dfrac{V_1-V_2}{V_2}$, and in the second cylinder is $\Pi \dfrac{\gamma}{\gamma-1}(V_1-V_2)$, Π being the atmospheric pressure. (*Dr Hopkinson.*)

80. A mass of liquid, in the form of an oblate spheroid, rotates uniformly about its axis; if the normal at any point P of the surface meet the axis in G, prove that the resulting attraction at g varies as PG.

If ϵ, the ellipticity, be supposed small, prove that, to a first approximation, $\omega^2 = \frac{16}{15}\pi\rho\epsilon$.

If, in the case when the eccentricity is very small, the mass be contracting slowly and uniformly and be so viscous that it takes up its positions of relative equilibrium instantaneously, prove that the ellipticity will vary as the cube root of the density.

81. If the Earth be completely covered by a sea of small depth, prove that the depth in latitude l is very nearly equal to $H(1-\epsilon\sin^2 l)$, where H is the depth at the equator, and ϵ the ellipticity of the Earth.

82. A mass (M) of homogeneous liquid revolves in relative equilibrium about a fixed axis with a uniform angular velocity such that the ellipticity (ϵ) of its surface is small. If the part μM of the mass were collected into an infinitely dense material point at the centre, and the density of the remaining part $(1-\mu)M$ were diminished in the ratio of $1-\mu$ to 1, find what would be the ellipticity of the new surface of equilibrium, supposing the time of rotation to be the same as before.

83. A solid ellipsoid of uniform density being supposed to revolve round its least axis of figure, and to carry with it a surrounding envelope of homogeneous liquid of different density, the entire mass attracting according to the law of nature, it is required to find the conditions requisite for the permanent assumption of the ellipsoidal form by the free surface. (*Prof. Townsend. Math. of Ed. Times*, Vol. XXXV.)

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